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Arbiter Assignment*

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Abstract

In dispute resolution, arbitrator assignments are decentralized and also incorporate parties' preferences, in total contrast to referee assignments in sports. We suggest that there can be gains (i) in dispute resolution from centralizing the allocation by bundling the newly arriving cases, and (ii) in sports from incorporating teams' preferences. To that end, we introduce a class of Arbiter Assignment Problems where a set of matches (e.g., disputes or games), each made up of two agents, are to be assigned arbiters (e.g., arbitrators or referees). On this domain, the question of how agents in a match should compromise becomes critical. To evaluate the value of an arbiter for a match, we introduce the (Rawlsian) notion of depth, defined as the arbiter's worst position in the two agents' rankings. Depth optimal assignments minimize depth over matches, and they are Pareto optimal. We first introduce and analyze depth optimal (and fair) mechanisms. We then propose and study strategy-proof mechanisms.

Keywords: Arbiter, arbitration, dispute resolution, assignment, mechanism, depth optimality, fairness, unanimity compromise, strategy-proofness, referee, sports, football.

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1 Introduction

"Every [referee] decision goes against us. In five games there have been none in our favour."[1]

The Merriam-Webster English Dictionary traces the term arbiter to the Latin root with the same spelling, meaning "eyewitness, onlooker, person appointed to settle a dispute." It defines an arbiter as "a person with power to decide a dispute" and lists its synonyms as "adjudicator, arbitrator, judge, referee, umpire".

Indeed, it is not hard to see many similarities between the setups where an arbitrator in dispute resolution and a referee in a sports event operate. Yet, dispute resolution and sports exhibit two key differences in the selection of arbiters. As will be detailed in the next section, in dispute resolution arbitrators are assigned to cases in a *decentralized* way that *takes into account the parties' preferences*. In sports, on the contrary, referees are assigned to matches in a *centralized* way that *does not take into account the teams' preferences*.

In this paper, we will suggest that the arbiter selection processes of these two setups can benefit from utilizing each other's key features. In dispute resolution, where cases are considered individually on a first-come first-served basis, there might be gains from bundling the newly arriving cases and using a centralized arbitrator assignment system instead. In sports, on the other hand, there are gains from incorporating teams' preferences in the referee assignment process. As exemplified in our opening quote, current practices more than rarely lead to frustration and disconcern, and mechanisms like the ones considered here will improve over them in terms of less controversy, leveling the playing field, transparency, fairness, and efficiency.

We will provide a general framework where there is a set of *matches* (e.g. disputes or games), each made up of two *agents* (e.g. parties to the dispute or contending teams) and each in need of an arbiter. *Our objective* is to utilize agents' preferences in a transparent and systematic way to assign arbiters to matches. Following a market design approach, we will solve this *Arbiter Assignment Problem* via mechanisms that produce desirable assignments.

As will be discussed in the next section, the two agents of a match may have distinct preferences on arbiters. Hence, while every match needs to be assigned a single arbiter, the choice of arbiter is a potential conflict that needs to be resolved among the agents of each match. Since the "how the agents should compromise" question becomes critical on one side of the market, our work introduces a brand new set of problems to the matching theory literature. As discussed in Section 2, our model is very different from even the seemingly closest "matching with couples" model since in the latter, each couple is assumed to have a single joint-preference over hospital pairs, hence how to evaluate assignments is not an issue.

A Brief Overview of Our Results:

To measure goodness of compromise, we define the *depth of an arbiter for a match* as the arbiter's worst ranking in the preferences of the agents in the match. For example, an arbiter

[1] The former Barcelona coach Ronald Koeman's complaints about the then-recent (February 2021) referee decisions against Barcelona after they lost two consecutive games (`dailymail.co.uk`).

that is second and third ranked by the two agents has a depth of three. Arbiters with smaller depth are more desirable compromises in a Rawlsian sense, that is, they make the worst-off agent better-off. Indeed, a well-known method to choose an arbiter for an isolated dispute is to minimize this depth. In the literature, this rule is known as the *Unanimity Compromise (UC)* (Hurwicz and Sertel, 1999). UC has several desirable properties. It is Pareto optimal and it picks an arbiter that each agent ranks as high as the median. A match can have at most two UC arbiters (Brams and Kilgour (2001)). Furthermore, these are "adjacent" in terms of ranking among efficient alternatives (Anbarci, 1993). UC has recently been adapted as a criterion for fairness by Barberà and Coelho (2022), who also note that UC is invariant with respect to undesirable candidacies (*i.e.*, adding arbiters considered to be worse by both agents than existing ones does not change the UC set).

For a centralized mechanism that produces multiple arbiter assignments, minimizing depth in every match might not be feasible. To achieve this as much as possible though, we define an arbiter assignment to be *depth optimal* if there is no other assignment that decreases the depth of some matches without hurting others. Depth optimality is one of the key concepts in this paper. We show that it implies Pareto optimality (but the converse is not true). Equally important, depth optimality also has a fairness aspect which, in every match, enables more desirable compromises in a Rawlsian sense. One of the main contributions of our paper is a class of *Depth Optimal Mechanisms (DOM)*. Informally speaking, a DOM follows an ordering of matches to identify an assignment that sequentially minimizes the depth for each match. Any multiplicity is solved through an additional ordering over arbiters. Each pair of orderings (one for matches, the other for arbiters) yields a different DOM. We first obtain that each DOM is depth optimal, hence Pareto optimal. Conversely, each depth optimal assignment can be obtained as a DOM outcome under some match-arbiter ordering.

Consider a potentially incomplete hierarchy of importance for matches (e.g., in sports, championship matches or matches among highly popular teams might have priority over other matches in the choice of referees, or in dispute resolution some cases with big total claims or involving a large number of workers for the union may have priority over other cases). Such a priority consideration would then necessitate a further property. To this end, we say that an assignment is *fair over matches* if there is no match where both agents prefer the arbiter assigned to a lower priority match over the arbiter assigned to theirs. The counterpart of this property in canonical matching problems, known as fairness, is quite central (e.g. see Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003)). While DOM fails to pass this test, a proper subclass of *Depth Optimal Priority Mechanisms (DOP)* are both fair over matches and depth optimal. In a DOP, match orderings are restricted to follow the matches' priorities.

Strategy-proofness requires a mechanism to provide incentives so that truthful reporting of preferences is in each agent's best interest. Hence, it is a highly desirable property. Unfortunately, there is a general tension between depth optimality and strategy-proofness. More specifically, no depth optimal mechanism - including DOM and DOP - is strategy-proof. Given this observation, one may wonder whether a weaker notion of compromise would be compatible with strategy-proofness. To that end, we say that an assignment is *minimally compromising* if no agent receives her least preferred arbiter. We find that no minimally compromising mechanism is strategy-proof. Hence, no compromise is possible under a strategy-proof mechanism.

Given the unavoidable tension between strategy-proofness and compromise among agents, we inquire whether any DOM is better than another in terms of manipulability. For this purpose, we compare the degree of manipulability of the DOM mechanisms *à la* Pathak and Sönmez (2013) and find that no DOM, hence DOP, is more manipulable than another.

In the second part of our paper, we focus on strategy-proof mechanisms. We introduce two mechanism classes: *Fairness-Corrected Serial Dictatorship (FSD)* and *Fairness-Corrected Sequential Dictatorship (FSED)*. Like DOM and DOP, these mechanisms treat matches sequentially.[2] But unlike them, FSD and $FSED$ pick in each match an agent, and follow her preferences when determining the arbiter for that match. Hence, in line with the above discussion, these mechanisms do not allow any compromise between the two agents of a match.

FSD and $FSED$ are strategy-proof and fair over matches. Hence, neither is depth optimal.[3] Furthermore, by utilizing Svensson (1999), we show that whenever each arbiter has a unit capacity, like in referee assignments, FSD is characterized by *one-sided responsiveness, non-bossiness*, and *neutrality* on top of fairness over matches and strategy-proofness. One-sided responsiveness ensures that in each match only one agent's preferences matter. Non-bossiness is a well-known property which requires that if an agent, by changing her preferences did not change her assigned arbiter, she also did not affect the arbiter assignment of other matches. Neutrality requires that the arbiters' identities do not matter for the assignment.

We also analyze the case of arbitrary capacities. We find that a mechanism is FSD if and only if it is *partially invariant to preferences (PIP), consistent*, fair over matches, and strategy-proof. PIP ensures that for each problem, there is a match-agent ordering such that no agent can change the earlier matches' assignments. Consistency is a well-known property which requires that a re-application of the mechanism after departure of a match along with its assigned arbiter-capacity does not change the assignment of the remaining matches. Like non-bossiness, consistency implies that an agent can only affect others' assignments through her own. We find that consistency characterizes the difference between FSD and $FSED$. That is, dropping consistency from the above set of axioms characterizes the larger class of $FSED$. We also establish the logical independence of these axioms.

2 Background

Dispute resolution is a major and growing sector, both at the international and the national levels. There are around $7,000$ arbitrators in the US alone and just the World Bank's International Center

[2] The ordering in a FSD is independent of the agents' preferences. The latter mechanism, $FSED$ is a generalization of FSD where who comes next in the order can depend on the preferences of the earlier agents.

[3] Thus, while there is incompatibility between depth optimality and strategy proofness, fairness over matches is compatible with both of them.

for the Settlement of Investment Disputes (ICSID) is composed of almost 500 arbitrators. The top three arbitral institutions in the US - the American Arbitration Agency (AAA), The Federal Mediation and Conciliation Service (FMCS), and the National Arbitration Board (NAB) - together handle a very large number of cases every year, with significant total claims. In 2020, just the AAA handled close to $10,000$ cases that in total involved over $18 billion.

Professional sports is also a multi-billion dollar business, involving teams, broadcasters and advertisers. The global sports market reached a value of nearly $458.8 billion in 2019. It is expected to reach $600 billion by 2025, and $826 billion by 2030 (`yahoo.com/news`). Every weekend, fans flock to stadiums or tune in to watch their favorite team play. The 2018 FIFA World Cup was watched by more than 3.5 billion viewers (`fifa.com/tournaments`).

Both dispute resolution and sports are sectors where arbiters play a significant role. However, the two setups do have some structural differences. In dispute resolution, arbitral institutions typically receive cases sequentially and, assign them to arbitrators in a first-come first-served basis. Additionally, cases are heterogenous in the amount of time needed to resolve them. In comparison, matches in a sports league are typically bunched together around, say, a weekend, and all matches have the same regulation duration. As a result, arbiter assignment has long been seen as a centralized allocation problem in sports, though not in arbitration. Another difference is that sometimes arbitrators can simultaneously handle multiple cases. However, a referee assigned to a match in a professional league is typically not assigned to another match that weekend.

Assigning Arbitrators in Dispute Resolution:

Arbitration is a central dispute resolution tool, both in the international and the national level. The ICSID mentioned above, the United Nations Commission on International Trade Law (UNCITRAL), and the International Chamber of Commerce (ICC) are some of the major arbitral institutions in the international arena and they mostly focus on international trade issues. At the national level, dispute resolution involves a range of issues such as contract arbitration, labor arbitration, securities arbitration, and judicial arbitration. For instance, the American Arbitration Association (AAA), the Federal Mediation and Conciliation Service (FMCS), and the National Arbitration Board (NAB) are some of the major arbitral institutions in the US.

At the international level, the longstanding principle is that it is up to the disputing parties to select the arbitrators. However, this practice has fueled a legitimacy debate, especially in international investment disputes where there can be prominent public interest (Penusliski, 2008; Paulsson, 2010) and states have been seeking to reassert control through limitations on the traditional party autonomy in selecting arbitrators, calling for more centralized mechanisms. Nevertheless, in the current practice the disputing parties can directly agree on an arbitrator.[4] If this fails, a backup method is used. For instance, ICSID uses a sequential method where the parties make alternating offers. UNCITRAL, on the other hand, uses a *Veto Rank* method where each party vetoes a number of alternatives from a list provided by UNCITRAL, and ranks the rest, and UNCITRAL picks

[4]If a three-person tribunal is to be assigned to a case, each disputing party nominates one party-appointed-arbitrator, and the presiding arbitrator is assigned by the arbitral institution respecting the involved parties' preferences.

among unvetoed alternatives the one with the best total-rank (Waibel, 2016). Arbitral institutions also recommend "informal mechanisms" to help the disputing parties select their arbitrators, the Veto Rank method mentioned above being the most common.[5]

At the national level, both AAA and FMCS in the US use versions of the Veto Rank method mentioned above. NAB, on the other hand, uses an *Alternate Strike* method where it provides the disputing parties with an initial list and asks them to alternately cross-off one name at each round. The last remaining name is chosen. There are other arbitral institutions that also use this method. Barberà and Coelho (2022) discusses the shortcomings of these alternatives and proposes to improve over them by mechanisms that implement the *Unanimity Compromise* rule of Hurwicz and Sertel (1999).

The literature has established that the parties in dispute can have systematically different preferences over arbitrators. For example, Bloom and Cavanagh (1986) shows that in labor disputes (i) both the employers and the unions tend to prefer individuals with law degrees to labor relations practitioners, but (ii) employers prefer economists to both of these groups whereas unions prefer both of these groups to economists.

The current assignment practices of arbitral institutions are far from satisfying desirable properties such as depth optimality and Pareto optimality. First, even in isolated cases rules like Veto Rank or Alternate Strike can be improved upon (De Clippel et al., 2014; Barberà and Coelho, 2022).[6]

Second, instead of treating each case individually, the arbitral institute can bundle cases that arrive in a predetermined time-window and use a centralized mechanism to assign arbitrators. This approach brings significant gains in kidney exchanges (Roth, Sönmez and Ünver, 2004). Example 4 in Section 4 elaborates on this point. As mentioned above, in 2021 the AAA received on average 30 cases per day. We argue that a centralized treatment of even weekly bundles of 200 cases can lead to remarkable welfare improvement. Given the significant length of the dispute resolution process, receiving a better arbitrator is worth the wait.[7]

Assigning Referees in Sports:

Following football terminology, we will use the term "referee" to describe the head official in charge of officiating a sports event. In other sports, a referee can be called, for example, a first official (volleyball) or a chair umpire (tennis). Referees play an important role in all sports. With the help of a team of deputies, they are in charge of enforcing the rules of the game.[8] Referee decisions may prove crucial for championship or relegation outcomes. They are a potential source

[5] De Clippel, Eliaz and Knight (2014) provides a formal and experimental critique of the Veto Rank method, and proposes an improvement called *Shortlisting*.

[6] In some stylized domains, it is even possible to construct Pareto optimal and strategy proof rules (e.g. see Kesten and Ozyurt (2021)).

[7] Even for the comparatively faster consumer arbitration cases, the average duration in AAA is 8 months (Horton and Chandrasekher, 2015). The duration is much longer in international arbitration. For example, in 2020 the average ICC case lasted 26 months and on average 3 new cases arrived every day (`ICC Statistics:2020`).

[8] In this paper, we assume that either a referee is assigned together with her deputies (as in football) or that the assignment of the referee is independent of the assignment of deputies (as in tennis). We, however, do not consider the problem of assigning multiple referees of equal hierarchy (as in basketball). This question is left for future research.

of controversy and frequently become headline news. Hence, for all established sports federations around the world, assigning referees to matches constitutes an important problem.

Many federations try to be very meticulous about referee assignments. In the UK Premier Football league for instance, referee assignments are made by the Professional Game Match Officials Board. The board takes into account several factors, including the referee's overall experience, current form, how often they have refereed the clubs involved, which team the referee supports, where they live, and any forthcoming international appointments (`premierleague.com`). They also consider the referee's past performance by evaluating decisions in previous matches via an elaborate point system. The Professional Game Match Officials Board however does not take into account the preferences of the involved teams.

The situation is no different in professional tennis. In every tournament, there is a "chief umpire" who assigns referees to matches. And as in football, tennis referees' overall experience and past performance affect assignment decisions. However, the players' preferences are not considered. These examples are no outliers. To the best of our knowledge, no sports federation takes into account teams' preferences when assigning referees to matches.

As suggested by the opening quote of our paper, teams on the other hand can have very strong preferences on referees. These preferences may be based on the team's beliefs about the referee's competence level as well as the referee's past decisions. After all, referees are human beings and at varying degrees, may be prone to the social pressure of the audience (Di Corrado, Pellarin and Agostini, 2011), may unintentionally favor the home team (Dohmen and Sauermann, 2016), or favor more successful teams (Erikstad and Johansen, 2020). A team might also have other reasons to think that a referee will be more or less lenient towards them.[9] Such preferences can also be context-dependent. And hence, not all information relevant to a team's preferences at a given time can be known by the federation. For example, a team with an upcoming international match might, in the domestic league, prefer a referee who is intolerant towards agression and hence, minimizes probability of injury for players.

Due to all these reasons, referee assignment practices that disregard teams' preferences are far from satisfying desirable properties such as depth optimality. As will be discussed later, our mechanisms allow teams to express their concerns/considerations through preference rankings and hence, help them avoid unwanted referees. Federations that respect these preferences achieve a normatively desirable compromise between teams, and reduce potential controversy. This might in turn improve the teams' morale and positively affect the quality of the game. In sports like basketball, teams' ratings of a referee's regular season performance affect the referee's promotion to playoffs. Hence, even though team preferences do not currently matter for referee assignments, performance evaluation based on team feedback is not uncommon. Such good practices can be further developed to incorporate team preferences in referee assignments.

Literature Review:

[9] For instance, in 2014 a former football referee, Eduardo Iturralde Gonzalez, made the following statement: "We don't come from Mars. You become a referee because you like football and there's no one that likes football that doesn't have a team" (`espn.com`).

The Arbiter Assignment Problem has not been previously analyzed. However, there are operations research and computer science papers that analyze scheduling problems in sports. These papers do not take into account the preferences of the teams. Instead, by taking into account a variety of constraints such as travel times or eligibility of referees, they formulate an optimization problem and analyze methods of solving it. Hence, central concepts in market design, such as strategy proofness or fairness do not play a role in this literature. For comprehensive reviews of such studies, please see Wright (2009) and Kendall, Knust, Ribeiro and Urrutia (2010).

Matching theory has found applications in several real-life markets, including worker-firm, student-school, donor-patient, and country-refugee matching (see Roth (1984), Abdulkadiroğlu and Sönmez (2003), Roth et al. (2004), and Ahani, Andersson, Martinello, Teytelboym and Trapp (2021)). This study adds to this growing literature by bringing a new problem to the table.

Theoretically, the novel aspect of our problem is that the arbiter assignment of a match affects both of its agents. More specifically, the arbiter is a public good that should be evaluated according to the preferences of both agents, and its choice requires compromise between the two. The "matching with couples" (Roth (1984)) model in the matching literature is the closest one to arbiter assignment. Particularly, there is a resemblance between the couples in that model and the matches in ours. However, there are important distinctions. First, the hospital seats in matching with couples are private goods. That is, the two doctors in a couple can be assigned to different hospitals. In arbiter assignment, however, both agents in a match should receive the same arbiter as a public good. Overall, matching with couples can be interpreted as a standard matching problem with externalities (because each agent's welfare is affected by her partner's assignment), but that is not the case for arbiter assignment. Another important difference is that, in matching with couples, each couple is already endowed with a joint preference relation over hospital pairs. How individual preferences in a couple should be aggregated to obtain that joint preference relation is outside the scope of that literature. On the other hand, in arbiter assignment, each agent is endowed with an individual preference relation. And, how the distinct individual preferences in each match should be aggregated to obtain a compromise is a central question. Another difference is that the matching with couples model is two-sided, that is, hospitals also have preferences over doctors. Arbiters in our model, however, do not have preferences over agents. In Section 6, we further discuss this assumption and analyze the implications of a two-sided version of our model. But even this two-sided version of our model is different from matching with couples since arbiters have preferences over matches, not over agents. Finally, due to the above disparities, the analyzed issues are also quite different. More specifically, a large bulk of the matching with couples literature deals with the existence of a stable matching.[10]

Our depth-optimal mechanisms – DOM and DOP – are reminiscent of the "priority mechanism", which is first used by Roth et al. (2004) to maximize the number of kidney exchanges. We utilize their idea to find an assignment that sequentially minimizes the depth for each match, and

[10] Roth (1984) shows that there does not always exist a stable matching. Klaus and Klijn (2005) find conditions over preferences to guarantee the existence; and Kojima, Pathak and Roth (2013) show that in large markets, existence is always obtained.

thus, obtain depth optimality.

As DOM is vulnerable to strategizing, we compare the degree of manipulability of DOM's à la Pathak and Sönmez (2013), which is the first to conduct a manipulability comparison analysis over matching mechanisms.

One of our central notions, depth optimality, is closely related to a choice rule called the Unanimity Compromise (UC), proposed by Hurwicz and Sertel (1999) and later analyzed by Brams and Kilgour (2001) and Kıbrıs and Sertel (2007) in the context of bargaining. Recently, Barberà and Coelho (2022) adapted UC as a criterion for fairness and studied game forms that implement it.[11] The Voting by Alternating Offers and Vetoes (VAOV) game form of Anbarci (1993) also implements UC.

This study also contributes to the literature on sequential and serial dictatorships (SD). Svensson (1999) was the first to axiomatize SD in a standard discrete unit-quota object allocation setting. Svensson and Larsson (2002) generalize this result to an object allocation problem with money. Pápai (2001) and Hatfield (2009) study SD in a model where agents have multi-unit demand. Afacan and Bo (2020) present a characterization based on "popularity of matchings". Pycia and Ünver (2020, 2021) study more general SD-based mechanisms.

3 Model

An **Arbiter Assignment Problem** consists of six parameters $(M, I, A, q, P, \succ)$, described below.

- $M = \{m_1, ..., m_n\}$ is the set of **matches**. We assume that $n \geq 2$.
- $I = \{i_{11}, i_{12}, ..., i_{n1}, i_{n2}\}$ is the set of **agents**. Each match m_k consists of two agents, the first agent i_{k1} and the second agent i_{k2}. For $M' \subseteq M$, $I(M')$ denotes the set of agents involved in the matches in M'. With an abuse of notation, we write $I(m)$ instead of $I(\{m\})$. Whenever it simplifies the exposition, we denote the first and the second agents in match m_k by $k1$ and $k2$, respectively.
- $A = \{a_1, ..., a_r\}$ is the set of available **arbiters**. Each arbiter a can be assigned a number of matches up to her capacity, denoted by q_a. Let $q = (q_a)_{a \in A}$ be the **capacity profile** of the arbiters. We assume that $\sum_{a \in A} q_a \geq max\{n, 3\}$.
- Each agent i has strict **preferences**, P_i, over arbiters.[12] R_i is the *at-least-as-good-as* relation associated with P_i.[13] $P = (P_i)_{i \in I}$ and $R = (R_i)_{i \in I}$ represent these preference profiles.
- Matches might be prioritized based on how critical they are. This **priority ordering** over matches is a transitive and asymmetric binary relation $\succ$ which need not be complete. When-

[11] The three game forms in Barberà and Coelho (2022) are based on a simple but important game form called "Rules of k Names" where one party offers a set of k alternatives and the other chooses one. Also see De Clippel et al. (2014), which theoretically and experimentally analyzes a member of this family.

[12] Formally, a strict preference is a complete, asymmetric, and transitive binary relation over A.

[13] That is, for any $a, a' \in A$, $a \; R_i \; a'$ if and only if $a = a'$ or $a \; P_i \; a'$.

ever $m \succ m'$, it means that m is more critical than m', hence has a higher priority than m'. We say that m and m' are in the same priority class and write $m \sim m'$ whenever neither $m \succ m'$ nor $m' \succ m$. An ordering of the matches – m'_1 ,..., m'_n – **respects** $\succ$ if $m'_k \succ m'_{k'}$ implies $k < k'$.

An *arbiter-assignment* (henceforth, **assignment**) μ is an allocation of arbiters over matches such that each match is assigned an arbiter, and no arbiter is assigned more matches than her capacity. We write $\mu(m_k)$ and $\mu(a)$ for the assignment of match m_k and arbiter a under assignment μ. Whenever an arbiter a is unassigned under μ, we write $\mu(a) = \emptyset$. Let $\mathcal{M}$ be the set of all assignments.

We define agents' preferences over assignments in a natural way: An agent ranks two assignments according to her preferences over the arbiters that she receives in each. Formally, given any pair of assignments μ, μ', any match $m \in M$ and $a \in I(m)$, agent a prefers μ to μ' if and only if $\mu(m)\ P_a\ \mu'(m)$. In our analysis M and q are allowed to vary. However, unless there is danger of confusion, we suppress all the primitives except the preferences from the notation and write P to denote a problem.

A **mechanism** ψ is a systematic way of producing an assignment at each problem. We write $\psi(P)$, $\psi_m(P)$, and $\psi_a(P)$ for its outcome at problem P and the assignments of match m and arbiter a, respectively.

The **depth** of arbiter a for match m_k is the worst ranking a receives from the two agents in m_k, according to their preferences $P_k = (P_{k1}, P_{k2})$. To formally define depth, let $\gamma(P_{ki}, a, A')$ represent the ranking of arbiter a in A' according to P_{ki}.[14] When $A' = A$, we suppress the last term and simply write $\gamma(P_{ki}, a)$. The depth of arbiter a for match m_k can now be formally written as

$$d_{m_k}(a|P_k) = \max\{\gamma(P_{k1}, a), \gamma(P_{k2}, a)\}.$$

The depth of an arbiter represents her ranking by the worst-off agent. Hence, minimizing depth amounts to maximizing social welfare by increasing the well-being of the worst-off agent in a match, in line with the seminal "egalitarianism principle" of Rawls (1971). The egalitarianism principle has strong fairness and efficiency properties. Therefore, it is not surprising that a well-known rule in the literature, called the **Unanimity Compromise rule (UC)** (Hurwicz and Sertel, 1999), maximizes egalitarian social welfare (Rawls, 1971) in every problem.[15] In our context, UC can be formally defined as follows. For a match m_k and a non-empty set of available arbiters $A' \subseteq A$, *UC* picks the arbiter(s) in A' with the lowest depth (equivalently, highest egalitarian social welfare)

[14] For instance, let P_{ki} : $a_1, a_2, a_3, ...,$ $A' = \{a_1, a_2, a_3\}$, and $A'' = \{a_2, a_3\}$. Then, $\gamma(P_{ki}, a_2, A') = 2$ while $\gamma(P_{ki}, a_2, A'') = 1$.

[15] The egalitarianism principle of Rawls (1971) and the utilitarianism principle of Jeremy Bentham constitute the two central ideas in social choice theory. Our depth notion is closely related to egalitarianism due to its emphasis on compromise, unlike utilitarianism. For example, if the two agents have diametrically opposed preferences, utilitarianism does not distinguish among the alternatives while egalitarianism picks the median alternative as a compromise. Alternatively, let the preferences be "almost" diametrically opposed as $P_1 : a_1, a_2, a_3, a_4, a_5$ and $P_2 : a_5, a_4, a_3, a_1, a_2$. Here, egalitarianism continues to favor compromise and recommends the median a_3. Instead, utilitarianism recommends a_1, where agent 1 makes no compromise at all.

according to the agents' preferences P_k:[16]

$$UC(m_k, P_k, A') = argmin_{a \in A'} d_{m_k}(a|P_k).$$

UC can alternatively be defined via an intuitive *Bargaining Algorithm* that highlights how it picks a compromise. Given the preferences (P_{k1}, P_{k2}) of the agents in match m_k,

Step 1 Each agent requests the top arbiter according to her ranking. If there is such an arbiter (that is, if the requests coincide and the arbiter is available), the algorithm chooses her and stops. Otherwise, the algorithm moves on to the next stage.

In general ...

Step k Each agent requests one of the top-k arbiters according to her ranking. If there are such arbiters, the algorithm chooses them all and stops. Otherwise, the algorithm moves on to the next stage.

UC satisfies a range of desirable properties (Kıbrıs and Sertel, 2007). First, it is both *anonymous* and *neutral*, that is, permuting the agents' preferences or the arbiters' names does not change the outcome. Hence, UC does not discriminate between agents or between arbiters, treating them all equally. Second, UC is *Pareto optimal*, that is, no alternative arbiter can be preferred by both agents to one assigned by UC. Third, UC is *monotonic*, that is, adding a new arbiter who is better for both agents than their worst-ranked UC arbiter cannot make an agent worse-off in comparison to his worst-ranked UC arbiter. Fourth, UC arbiters are *connected*, that is, there can not be a third arbiter ranked by both agents in between two UC arbiters. This is because, in this case this third arbiter would be a better compromise and would have been picked by UC instead. Finally, UC satisfies a weaker version of Maskin monotonicity (Maskin, 1985) restricted to preference changes that preserve "symmetry" (Kıbrıs and Sertel (2007)).

The example below demonstrates how the two definitions of UC can be used to pick an arbiter.

Example 1. ***An application of UC*** *Let $A = \{a_1, ..., a_5\}$ be the set of all arbiters and let $A' = \{a_1, a_2\}$ be the set of available arbiters for match m_k. Let the agents' preferences be:*

P_{k1}	a_3	a_1	a_2	a_5	a_4
P_{k2}	a_2	a_3	a_5	a_1	a_4

In this case,

$$UC(m_k, P_k, A') = argmin_{a \in A'} d_{m_k}(a|P_k) = \{a_2\}$$

since the depths of the arbiters a_1, a_2, a_3, a_4, a_5 are $4, 3, 2, 5, 4$, respectively. Note that while a_3 has the lowest depth, this arbiter is not available. Hence, UC picks a_2. This arbiter can alternatively be chosen by applying the Bargaining Algorithm as follows.

1. *Agents 1 and 2 request a_3 and a_2, respectively. Their requests are not compatible since they do not contain a common arbiter.*

[16]For ease of notation, we suppress the preferences notation from UC whenever there is no danger of confusion.

2. *Agents 1 and 2 request $\{a_3, a_1\}$ and $\{a_2, a_3\}$, respectively. While their requests contain a common arbiter, a_3, this arbiter is not available.*

3. *Agents 1 and 2 request $\{a_3, a_1, a_2\}$ and $\{a_2, a_3, a_5\}$, respectively. The arbiter a_2 is common to both requests and is also available. Hence, the algorithm stops and picks a_2.*

When there are multiple matches to be considered, simultaneously minimizing depth by applying UC to every match might not be feasible. For example, consider two matches and two arbiters, each with capacity 1. Assume that all agents prefer arbiter a to a'. In this case, minimizing depth in both matches is not feasible since a can be assigned to only one match. Therefore, to pursue the objective of minimizing depth as much as possible, we introduce the following definition. An assignment μ is **depth optimal** if there is no assignment μ' such that for each $k \in \{1, ..., n\}$, $d_{m_k}(\mu'(m_k)|P) \leq d_{m_k}(\mu(m_k)|P)$, with strict inequality for some k. That is, an assignment is depth optimal if it is impossible to find a reassignment that decreases the depth in one match without increasing that of another's. Depth optimality can be interpreted as a "Pareto optimality" property in an environment where the matches are treated as "agents" and "preferences of a match" are obtained by aggregating preferences over its agents using the notion of depth. Therefore, the set of depth optimal assignments contains all alternative ways of simultaneously minimizing depth in all matches to the extent possible. Furthermore, as will be discussed in the next section, sequential applications of UC over matches can achieve all depth optimal assignments. Hence, all desirable properties of UC discussed above apply to depth optimal assignments. Particularly, the set of depth optimal assignments is *anonymous* and *neutral*. That is, it does not discriminate among agents, matches, or arbiters. Depth optimal assignments are also *monotonic*, that is, starting from a depth optimal assignment, addition of an arbiter that decreases depth in one match does not increase depth in other matches. Depth optimal assignments are also *Pareto optimal*, as will be discussed next.

Assignment μ is **Pareto optimal** if there is no other assignment μ' such that for each agent $i \in I$, $\mu' \; R_i \; \mu$, where this strictly holds for some agent. The following proposition shows that depth optimality has a clear efficiency implication. All the proofs are relegated to the Appendix.

Proposition 1. *If an assignment μ is depth optimal for a problem P, then it is also Pareto optimal for P. Yet, the converse is not true.*

We next analyze an adaptation to our domain of the central fairness property in market design. This property, commonly referred to as "fairness" in that literature, requires that higher priority agents do not envy those with lower priority.[17] Since, in our domain, there are no priority differences between agents, fairness has no direct implication. However, there are arbiter assignment problems where some matches have higher priority over others. For example, the policymaker might want dispute cases with high public visibility to be of higher priority than others. Similarly, games

[17] Note that "fairness" is a weakening of one of the central fairness properties in economic design called "no-envy" (Foley, 1966). Similar formulations that treat agents asymmetrically exist on other domains as well (*e.g.* see the "hierarchical no-envy" property in Kıbrıs (2003)).

that are critical for championship in a tournament are considered to have higher priority. Our formulation requires respect for such priority differences between matches. Formally, we say that an assignment μ is **fair over matches** if there is no pair of matches m, m' such that $m \succ m'$, and for each agent $i \in I(m)$, $\mu(m')\ P_i\ \mu(m)$. In words, fairness over matches requires that the agents of a higher priority match do not unanimously prefer the arbiter of a lower priority match to theirs. Recall that $\succ$ does not have to be complete, allowing for incomparable matches based on $\succ$.

A mechanism ψ is <*Pareto optimal*, *depth optimal*, *fair over matches*> if, at each problem P, $\psi(P)$ is <Pareto optimal, depth optimal, fair over matches>.

4 Depth Optimal Mechanisms

In this section, we propose a class of mechanisms. Each mechanism utilizes two separate orderings, one for the matches and one for the arbiters. By following the match ordering, the algorithm iteratively calculates each match's UC arbiters among those that are available to it. Whenever a match has only one UC arbiter, that arbiter is assigned to the match. If it has two, then both of these arbiters are tentatively assigned to the match until another match admits either of them as its unique UC arbiter, in which case the remaining arbiter is assigned to this match. If this does not happen, then the match receives among the two tentatively assigned arbiters the one with the lowest index.[18] At each step, the assigned matches and arbiters are removed from the pool, and the algorithm continues working in the same manner in the reduced problem.

Members of this class of mechanisms always produce depth-optimal assignments. And, as mentioned in the previous paragraph, they only differ from each other in the match and the arbiter orderings that they employ as a function of the priority ordering $\succ$. To this end, let Δ be a function that maps each $\succ$ to a pair of orderings $M = \{m_1, .., m_n\}$ and $A = \{a_1, .., a_r\}$.[19] Given a problem P, this mechanism uses the following algorithm: for $t \geq 1$,

Step t.1 Let $M^t = \{m_\ell, ..., m_{\ell'}\}$ and $A^t = \{a_j, ..., a_{j'}\}$ be the sets of matches and arbiters, each written in order of the original enumeration (that is, those with lower indexes come earlier). Let $\mathcal{M}^t$ be the set of feasible assignments at Step t. For $t = 1$, let $M^1 = M$, $A^1 = A$, and $\mathcal{M}^1 = \mathcal{M}$.

Step t.2 We run the following steps one by one following the ordering of matches.

Substep t.2.1 Let $\epsilon^1 = \{\mu \in \mathcal{M}^t :\ \mu(m_\ell) \in UC(m_\ell, A^t)\}$.

In general, for $k \leq \ell'$,

Substep t.2.k

$$\epsilon^k = \begin{cases} \{\mu \in \epsilon^{k-1} :\ \mu(m_k) \in UC(m_k, A^t)\} & \text{if } \mu(m_k) \in UC(m_k, A^t) \text{ for some } \mu \in \epsilon^{k-1}, \\ \epsilon^{k-1} & \text{otherwise.} \end{cases}$$

[18] Note that, at any stage of the algorithm there might be multiple such matches. For example if matches m_1, m_2, m_3 are tentatively assigned to arbiters $\{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}$, respectively, then each m_i receives a_i. Alternatively, if these matches are tentatively assigned to $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}$, respectively, then once one of the arbiters, say a_1 is assigned to a later match each match m_i receives a_{i+1}.

[19] Note that the ordering of matches $M = \{m_1, .., m_n\}$ need not respect $\succ$.

Step $t.2$ terminates by the end of Substep $t.2.\ell'$, producing $\epsilon^{\ell'}$. By construction, $\epsilon^{\ell'}$ cannot be empty. Let $M' \subseteq M^t$ be the set of matches that receive their UC arbiters at this step, that is, such that for each $m \in M'$ and $\mu \in \epsilon^{\ell'}$, $\mu(m) \in UC(m, A^t)$.[20] Following the ordering of matches one by one, let each match $m \in M'$ receive the lowest indexed unassigned arbiter among those it is temporarily assigned in $\epsilon^{\ell'}$. These assignments become final, and we remove the matches which received an arbiter and write M^{t+1} for the set of remaining matches. We also decrease the capacity of each arbiter by the number of matches she is assigned. Let A^{t+1} be the set of arbiters that still have a positive capacity. Let $\mathcal{M}^{t+1}$ be the set of assignments in the reduced problem.

If at the end of Step t no match is left unassigned, then the algorithm ends. Otherwise, we run the same steps above for $t+1$. In each iteration, at least one match is assigned an arbiter and is removed from the problem. This, as well as the fact that there are finitely many matches, shows that the algorithm terminates in finitely many steps. The arbiter assignment, obtained in the course of the above process, constitutes the algorithm's final outcome.

The above algorithm defines what we call the **Depth Optimal Mechanism (DOM)**.[21] The following examples demonstrate how the DOM algorithm proceeds to obtain an assignment.

Example 2. ***An application of DOM*** *Let $M = \{m_1, .., m_4\}$ and $A = \{a_1, ..., a_4\}$, with $q_a = 1$ for each $a \in A$. Let us consider the DOM whose match and arbiter orderings follow the matches' and arbiters' indexes. Let the preferences be as follows:*

P_{11}	P_{12}	P_{21}	P_{22}	P_{31}	P_{32}	P_{41}	P_{42}
a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2
a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1
a_3	a_3	a_3	a_3	a_3	a_3	a_3	a_3
a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4

The DOM algorithm works as follows.

1. *ϵ^1 is the set of all μ such that $\mu(m_1) \in \{a_1, a_2\}$. That is, $UC(m_1, A) = \{a_1, a_2\}$.*

2. *ϵ^2 is the subset of ϵ^1 where $\mu(m_2) \in \{a_1, a_2\}$. That is, $UC(m_2, A) = \{a_1, a_2\}$.*

3. *Since $UC(m_3, A) = UC(m_4, A) = \{a_1, a_2\}$, and these arbiters are already assigned under ϵ^2, we have $\epsilon^4 = \epsilon^3 = \epsilon^2$ (recall that $q_a = 1$ for each $a \in A$).*

   Due to the tie-breaking rule, m_1 and m_2 respectively receive a_1 and a_2. We remove these matches and decrease the capacity of the assigned arbiters by one, making their reduced capacities zero. Hence, $M^2 = \{m_3, m_4\}$ and $A^2 = \{a_3, a_4\}$. We then continue applying the same steps for these sets of matches and arbiters.

[20] Note that if for a match m, $\mu(m) \in UC(m, A^t)$ for some $\mu \in \epsilon^{\ell'}$, then for any assignment $\mu' \in \epsilon^{\ell'}$, we have $\mu'(m) \in UC(m, A^t)$.

[21] This mechanism is motivated by the priority mechanism, which, to the best of our knowledge, was first introduced for kidney exchange problems by Roth, Sönmez and Ünver (2005).

4. *If we apply Step 2 of DOM for match m_3 and write ϵ'^1 for the associated set, then it consists of assignments where match m_3 receives a_3. That is, $UC(m_3, \{a_3, a_4\}) = \{a_3\}$.*

5. *Since $UC(m_4, \{a_3, a_4\}) = \{a_3\}$, match m_4 can not receive its lowest depth arbiter (in the reduced problem) at any assignment in ϵ'^1. Hence, We have $\epsilon'^2 = \epsilon'^1$. Therefore, this step terminates here. We remove match m_3 along with its assignment a_3. We decrease the capacity of a_3 by one making its reduced capacity zero. Hence, we have $M^3 = \{m_4\}$ and $A^3 = \{a_4\}$.*

6. *The only arbiter left to match m_4 is a_4. Hence, it is assigned to a_4, and the mechanism terminates. If we write μ for the outcome, then $\mu(m_1) = a_1$, $\mu(m_2) = a_2$, $\mu(m_3) = a_3$, and $\mu(m_4) = a_4$.*

In some problems, the tentative assignments in the DOM algorithm might not clear as fast as in the previous example. The following example demonstrates this point.

Example 3. ***Another application of DOM*** *Let $M = \{m_1, .., m_4\}$ and $A = \{a_1, ..., a_4\}$, with $q_a = 1$ for each $a \in A$. Again let us consider the DOM whose match and arbiter orderings follow the matches' and arbiters' indexes. Let us consider the following preferences below:*

P_{11}	P_{12}	P_{21}	P_{22}	P_{31}	P_{32}	P_{41}	P_{42}
a_1	a_2	a_1	a_3	a_3	a_4	a_3	a_3
a_2	a_1	a_3	a_1	a_4	a_3	a_4	a_4
a_3	a_3	a_2	a_2	a_5	a_5	a_1	a_1
a_4	a_4	a_5	a_5	a_1	a_1	a_5	a_2
a_5	a_5	a_4	a_4	a_2	a_2	a_2	a_5

1. *ϵ^1 is the set of all μ such that $\mu(m_1) \in \{a_1, a_2\}$. That is, $UC(m_1, A) = \{a_1, a_2\}$.*

2. *ϵ^2 is the subset of ϵ^1 where $\mu(m_2) \in \{a_1, a_3\}$. That is, $UC(m_2, A) = \{a_1, a_3\}$.*

3. *ϵ^3 is the subset of ϵ^2 where $\mu(m_3) \in \{a_3, a_4\}$. That is, $UC(m_3, A) = \{a_3, a_4\}$.*

4. *ϵ^4 is the subset of ϵ^3 where $\mu(m_4) = a_3$. That is, $UC(m_4, A) = \{a_3\}$.*

   The assignment $\mu(m_4) = a_3$ creates a chain reaction where $\mu(m_3) = a_4$, $\mu(m_2) = a_1$, and finally $\mu(m_1) = a_2$. Since all assignments are finalized, the mechanism terminates.

The reader might wonder whether the way DOM mechanisms handle multiple UC arbiters (as demonstrated in the above examples) is unnecessarily complicated and whether simpler methods exist. A particularly salient suggestion would be to exogenously designate an agent in each match and to let her choose one of the two UC arbiters (such as letting the visiting team make the choice in sports). However, such a mechanism would violate depth optimality. To see this, note that letting the second agent in every match to choose from UC arbiters would yield $\mu'(m_1) = a_2$, $\mu'(m_2) = a_3$, $\mu'(m_3) = a_4$, and $\mu'(m_4) = a_1$, and this assignment violates depth optimality.

Since changing Δ (that is, the ordering of arbiters and matches as a function of $\succ$) also changes the way the above algorithm works, each Δ defines a DOM. Spanned over all possible Δ, the above definition specifies our mechanism class. The results stated below holds for any DOM, that is, for any Δ.

Theorem 1. *Every depth optimal assignment can be obtained as the outcome of a DOM under some Δ. Conversely, every DOM produces depth optimal and Pareto optimal assignments.*

Remark 1. *It is useful to mention that in applications to sports the numbers of matches and arbitors are not forbiddingly large for a practical application of DOM. For example, in a typical professional football league, each week there are around* 10 *matches and* 20 *referees.*[22] *In dispute resolution, the arbitral institutions have control over the numbers of matches and arbitors. By choosing these numbers appropriately, they can ensure that DOM is practically feasible and still improve over decentralized assignments (see Example 4 below).*

It is easy to verify that not every DOM is fair over matches. For instance, consider the DOM in Example 3 when m_4 has the highest priority: $m_4 \succ m_k$ $(k < 4)$.

Fact 1. *Not every DOM is fair over matches.*

Fortunately, we can easily define a subclass of DOM whose members are fair over matches. Consider each DOM whose match orderings always respect $\succ$. We call such DOM a **Depth Optimal Priority Mechanism (DOP)**.

Proposition 2. *Every DOP is depth optimal, Pareto optimal, and fair over matches.*

Remark 2. *Since a DOP must respect match priorities, not every depth optimal assignment that is fair over matches can be obtained as the outcome of a DOP. For instance, let $M = \{m_1, m_2\}$ with $m_1 \succ m_2$, and $A = \{a_1, a_2, a_3\}$ with $q_1 = q_2 = q_3 = 1$. Let $P_{11}: \ a_1, a_2, a_3$; $P_{12}: \ a_3, a_2, a_1$; and $P_{21} = P_{22}: \ a_2, a_3, a_1$. Let μ be an assignment where $\mu(m_1) = a_1$ and $\mu(m_2) = a_2$. Note that μ is both depth optimal and fair over matches. However, it cannot be obtained through a DOP. This is because the only match ordering that respects $\succ$ is m_1, m_2. Under this ordering, the DOP outcome, independent of the arbiter ordering, is μ' where $\mu'(m_1) = a_2$ and $\mu'(m_2) = a_3$.*

It is obvious that DOM (hence, DOP) improves over the current assignments in markets where the agents' preferences are ignored, such as in referee assignments in sports. It might be less obvious to see that these mechanisms also improve over arbiter assignments where the agents' preferences are respected but the cases are individually treated on a first-come first-served basis rather than being handled together (even though in each individual case, depth optimality might have been respected). The following example demonstrates this point. It shows that DOP leads to an improvement in terms of both depth optimality and Pareto optimality over the current practices of arbitral institutions.

[22] For the English Premier League and the Spanish La Liga, see (`premierleague.com`) and (`sportsmole.co.uk`).

Example 4. *Let $M = \{m_1, m_2\}$ and $A = \{a_1, a_2\}$, each with unit capacity. Let P be such that $P_{11} = P_{12} :\ a_2, a_1$; and $P_{21} = P_{22} :\ a_1, a_2$. Suppose m_1 and a_1 arrive first (that is, only a_1 is available by the arrival of m_1). Arbiter a_1 is assigned to m_1 without waiting for the arrival of m_2 and a_2. After match m_2 and arbiter a_2 arrive, m_2 receives a_2. The outcome is $\mu(m_1) = a_1$ and $\mu(m_2) = a_2$. On the other hand, if we pool the matches and the arbiters and run the DOP, we obtain the assignment $\mu'(m_1) = a_2$ and $\mu'(m_2) = a_1$. Note that μ' improves upon μ in terms of both Pareto optimality and depth optimality. It is also fair over matches independent of the matches' priorities.*

We next turn to strategic considerations. A mechanism ψ is **manipulable at a problem** P if there is an agent i with P'_i such that $\psi(P'_i, P_{-i})\ P_i\ \psi(P)$.[23] A mechanism ψ is **strategy-proof** if it is not manipulable at any problem.

Unfortunately, no DOM, hence no DOP, is strategy-proof. However, this problem is not specific to DOM. As the following result shows, there is a general tension between depth optimality and strategy-proofness.

Proposition 3. *No mechanism is both depth optimal and strategy-proof.*

Corollary 1. *No DOM, hence no DOP, is strategy-proof.*

Depth optimality implies Pareto optimality as well as a compromise between the two agents in the same match. The latter clashes with strategy-proofness. Thus, one may wonder whether a weaker notion of "compromise between agents" would be compatible with strategy proofness. To this end, consider the **minimally compromising set** of assignments, that is, assignments where no agent receives its least preferred arbiter (recall that $|A| = r$):

$$MC(P) = \{\mu \in \mathcal{M} : \text{ for each match } m \text{ and } i \in I(m), \gamma(P_i, \mu(m)) < r\}.$$

Note that $MC(P)$ can be empty. A mechanism ψ is **minimally compromising** if, at each problem P, $\psi(P) \in MC(P)$ whenever $MC(P) \neq \emptyset$. Unfortunately, our next result establishes that even this minimal compromise is not possible under a strategy-proof mechanism.

Proposition 4. *No mechanism is both strategy-proof and minimally compromising.*

Given the severe tension between compromising and strategy-proofness, we next compare the degree of manipulability of the DOM mechanisms. The following notion of manipulability comparison is taken from Pathak and Sönmez (2013). A mechanism ψ is **more manipulable than** ϕ if (i) whenever ϕ is manipulable at a problem, then so is ψ, and (ii) at some problem, ϕ is not manipulable, yet ψ is.[24]

[23] P_{-i} is the preference profile of all agents but agent i.

[24] For many-to-many matching problems, Chen, Egesdal, Pycia and Yenmez (2016) introduce an "agent level" manipulability comparison: for an agent i, ψ is more manipulable than ϕ if the set of preferred assignments agent i receives by misreporting her preferences under ψ is never a subset of that under ϕ, and the opposite holds at some problem. We do not pursue this notion in this paper.

Theorem 2. *No DOM or DOP is more manipulable than another DOM.*

These results tell us that each DOM, hence DOP, is equally manipulable. Hence, in terms of the degree of manipulability, there is no reason to choose one over another.

5 Fairness-Corrected Serial and Sequential Dictatorships

In this section, we introduce two strategy-proof mechanisms: Fairness-Corrected Serial Dictatorship (FSD) and Fairness-Corrected Sequential Dictatorship ($FSED$).

5.1 Fairness-Corrected Serial Dictatorship

Let Γ be a function that maps each $\succ$ to an ordering of match-agent pairs $\Gamma_{\succ} = \{(m_1, i_1), ..., (m_n, i_n)\}$ where for each $k \leq n$, $i_k \in I(m_k)$. We assume that Γ respects $\succ$, that is for each $\succ$, $m_k \succ m_{k'}$ implies $k < k'$.[25] We are now ready to define FSD. Given a problem P,

Step 1. We consider match m_1 under $\Gamma_{\succ}$. We assign the top arbiter of agent i_1 to match m_1. We remove the match and decrease the capacity of its assigned arbiter by one.

In general,

Step k. We consider match m_k under $\Gamma_{\succ}$. Among the arbiters with a positive reduced capacity, we assign the top arbiter of agent i_k to match m_k. We remove the match and decrease the capacity of its assigned arbiter by one.

This algorithm terminates by the end of Step n, by which each match is assigned an arbiter. It defines what we call the **Fairness-Corrected Serial Dictatorship Mechanism (FSD)**. Since changing Γ also changes the way the above algorithm works, each Γ defines a different FSD. Hence, the above definition specifies a class of FSD mechanisms. Note that the above procedure where Γ does not necessarily follow $\succ$ defines a larger class of mechanisms, which is the straightforward extension of the usual serial dictatorship (SD) to our problem.

5.2 Fairness-Corrected Sequential Dictatorship

For any $\succ$, let $\pi_{\succ}$ be a rule that determines the next match along with an agent in the match as a function of the preferences of the earlier matches' agents. Note that $\pi_{\succ}$ specifies the first-match and the first-agent independent of the preferences. Let π be function that maps each $\succ$ to a rule $\pi_{\succ}$. We assume that π respects $\succ$, that is for each $\succ$ and P, $m \succ m'$ implies that m comes earlier than m' under $\pi_{\succ}$. We are now ready to define $FSED$. Given a problem P,

Step 1. Let $\pi_{\succ}$ decide the first match-agent pair. We assign the top arbiter of the agent to the match. We then remove the match and decrease the capacity of the assigned arbiter by one.

In general,

[25] We also assume that for any $\succ$, Γ preserves the relative match-agent orderings in the reduced problems.

Step k. Let $\pi_{\succ}$ decide the next match-agent pair. Among the arbiters with a positive reduced capacity, we assign the top arbiter of the agent to the match. We then remove the match and decrease the capacity of the assigned arbiter by one.

This algorithm terminates by the end of Step n, by which each match is assigned an arbiter. The algorithm defines what we call the **Fairness-Corrected Sequential Dictatorship Mechanism (FSED)**. Since changing π also changes the way the above algorithm works, each π defines a different $FSED$. Note that FSD is a proper subclass of $FSED$. To highlight the difference, we present in Appendix D a $FSED$ mechanism which is not FSD.

Proposition 5. *Each $FSED$, hence each FSD, is strategy-proof and fair over matches.*

The following corollary follows from Proposition 3 and Proposition 5.

Corollary 2. *No $FSED$, hence no FSD, is depth optimal.*

5.3 Characterizations of FSD and FSED

In what follows, we provide axiomatic characterizations of FSD and $FSED$. The following axiom requires that in each match, preferences of only one agent matter in determination of its assigned arbiter. A mechanism ψ is **one-sided responsive** if for each match m there is an agent $i \in I(m)$ such that $\psi_m(P) = \psi_m(P')$ whenever $P_i = P'_i$. Next is a well-known axiom which requires that the resulting assignment should not be affected if an agent changing her preferences does not affect her own arbiter. A mechanism ψ is **non-bossy** if for each problem P, match m, and agent $i \in I(m)$ with P'_i, whenever $\psi_m(P'_i, P_{-i}) = \psi_m(P)$, we have $\psi(P) = \psi(P'_i, P_{-i})$. The following observation follows from the definitions of FSD and $FSED$.

Fact 2. *Each $FSED$, hence each FSD, is one-sided responsive. While each FSD is non-bossy, not every $FSED$ is non-bossy.*

The following lemma will be critical for the first main result of this section. It states that whenever a mechanism is one-sided responsive and non-bossy, its outcome only depends on the preferences of a particular set of agents, including only one agent from each match.

Lemma 1. *Let ψ be a one-sided responsive and non-bossy mechanism. Then, there is a set of agents $I' = \{i_1, ... i_n\}$, where $i_k \in I(m_k)$ for each $k \leq n$, such that for each P and P' with $(P_i)_{i \in I'} = (P'_i)_{i \in I'}$, $\psi(P) = \psi(P')$.*

In other words, we can consider each agent from this set as the sole representation of her match, and any one-sided and non-bossy mechanism needs to assign these agents (or equivalently their matches) to arbiters only based on their preferences. Thus, with these two axioms, the mechanism class that we consider reduces to one in standard matching problems. This, in turn, enables us to utilize the existing SD characterizations. Particularly, Svensson (1999) obtains that whenever each object has unit quota, a mechanism is SD if and only if it is non-bossy, strategy-proof, and *neutral*.[26] Together with Lemma 1, this result yields the following characterization for the special case of unit capacity arbiters, as in the case of referee assignments. As shown in the Appendix, these axioms are logically independent.

Corollary 3. *Suppose the capacities are fixed to $q_a = 1$ for each $a \in A$. A mechanism is a FSD if and only if it is one-sided responsive, non-bossy, fair over matches, strategy-proof, and neutral.*

We now turn to the more general problem of arbitrary capacities. Our first axiom is consistency. This well-known axiom requires that removing an assigned match along with its arbiter-capacity does not alter the assignment of the other matches. Formally, for $M' \subseteq M$ and q' where $q'_a \leq q_a$ for each $a \in A$, let $\psi(P|M', q')$ denote the outcome of ψ in a problem consisting of matches M' and arbiters A with the capacity profile of q'. (Whenever $M' = M$ and $q' = q$, we suppress the notation and just write $\psi(P)$ as before.) A mechanism ψ is **consistent** if for each P and match m with $\psi_m(P) = a$, $\psi_{m'}(P|M \setminus \{m\}, q') = \psi_{m'}(P)$ for each $m' \neq m$, where $q' = (q_a - 1, q_{-a})$.[27]

Our second axiom limits how mechanism reacts to a change in the agents' preferences. It states that for each problem there is a match-agent ordering such that the agents' preferences do not affect the assignments of the earlier matches. A mechanism ψ is **partially invariant to preferences (PIP)** if for each problem P, there is a match-agent ordering $\Upsilon = \{(m_1, i_1), ..., (m_n, i_n)\}$[28] such that for each i_k and P'_{i_k}, (i) $\psi_{m_{k'}}(P) = \psi_{m_{k'}}(P'_{i_k}, P_{-i_k})$ for each $k' < k$, (ii) $\gamma(P'_{i_k}, \psi_{m_k}(P'_{i_k}, P_{-i_k}), A') = \gamma(P_{i_k}, \psi_{m_k}(P), A')$ where $A' = A \setminus \{a \in A : \ q_a = |\{k' < k : \ \psi_{m_{k'}}(P) = a\}|\}$, and (iii) for $j \in I(m_k) \setminus \{i_k\}$ and each P'_j, $\psi_{m_{k'}}(P) = \psi_{m_{k'}}(P'_j, P_{-j})$ for each $k' \leq k$.

Theorem 3. *A mechanism is consistent, PIP, fair over matches, and strategy-proof if and only if it is a FSD.*

The following theorem shows that when consistency is dropped from the list of axioms in Theorem 3, the class of $FSED$ is obtained.

Theorem 4. *A mechanism is PIP, fair over matches, and strategy-proof if and only if it is a $FSED$.*

The independence of the axioms in Theorem 3 and Theorem 4 is provided in the Appendix.

6 Implications of a Two-Sided Model

As discussed in Section 2, in current real-life applications of arbiter assignment it is standard practice to disregard arbiters' preferences over matches. In case of sports, an extensive literature establishes several potential biases a referee might suffer from. This leads federations to not only disregard referees' preferences over matches but to take great measures (such as using information on

[26] Neutrality requires that the objects' identities do not matter for the outcome. Its formal definition is provided in the Appendix.

[27] q_{-a} is the capacity profile of the arbiters $A \setminus \{a\}$ under q.

[28] Note that Υ does necessarily respect $\succ$.

where the referee is born and where they live and so on) to eliminate any factor that might affect referee judgements and create suspicions of impartiality. Similarly, in case of dispute resolution Section 2 lists a range of alternative rules that are either used in real life or proposed by the recent theoretical literature. A common feature of all of these rules is that they all disregard arbitrators' potential preferences over cases, treating arbitrators as objects.[29] Overall, in both applications the arbiters have a significantly different role than the agents. While following her incentives (*e.g.* to win a dispute or a game) is an agent's prerogative, the arbiter is a service provider whose main duty is to be impartial. Hence, the common practice is to treat arbiters as objects.

Due to the above reasons, in this paper we analyze a one-sided assignment problem where arbiters do not declare preferences over matches. However, it is still of theoretical interest to see what happens if arbiter preferences are also included in the model. Moreover, there might be other applications where taking arbiter preferences into account would be acceptable. Hence, in this section we discuss a two-sided version of our model where the arbiters are also endowed with preferences over matches.

Throughout this section, assume on top of our standard model that each arbiter a has preferences P_a over matches $M \cup \{\emptyset\}$, where $\emptyset$ represents being unassigned. Let $P = (P_i)_{i \in I \cup A}$ be the preference profile of all agents and arbiters. First of all, as each match must receive an arbiter, it is impossible to guarantee that no arbiter is ever assigned to a match that she ranks under being unassigned. Formally speaking, individual rationality cannot be guaranteed for arbiters. Hence, it is outside the scope of our analysis. On the other hand, as both sides have preferences, blocking is a relevant issue. Formally, an assignment μ is **blocked** if there is a match-arbiter pair (m_k, a) such that $m_k \; P_a \; \mu(a)$, $a \; P_{k1} \; \mu(m_k)$, and $a \; P_{k2} \; \mu(m_k)$. Assignment μ is **stable** if it not blocked.

We first observe that depth optimality is incompatible with stability. To see this, consider three matches m_1, m_2, m_3 and three arbiters a_1, a_2, a_3, each with a unit quota. Let the agents' preferences be such that $P_{11} = P_{12} : \; a_1, a_2, a_3$; $P_{21} = P_{22} : \; a_2, a_1, a_3$; $P_{31} = P_{32} : \; a_2, a_3, a_1$. Let the arbiters' preferences be such that $P_{a_1}; \; m_2, ..$; $P_{a_2} : \; m_1, m_3, ...$ The unique stable assignment, say μ, is such that $\mu(m_1) = a_2$, $\mu(m_2) = a_1$, and $\mu(m_3) = a_3$. This assignment is not depth optimal, however, as one can improve its depth profile by swapping the assignments of m_1 and m_2. This finding also implies that no DOM, hence DOP, is stable.

A second observation is that stability is incompatible with fairness over matches. Furthermore, the two requirements can clash even in problems where a stable and depth optimal assignment does exist. To see this, let us consider two matches m_1, m_2, and two arbiters a_1, a_2, each with a unit quota. Suppose that $m_1 \succ m_2$. Let all the agents have the same preferences where the top arbiter is a_1. On the arbiter-side, suppose a_1 prefers m_2 to m_1. The unique stable assignment μ is such that $\mu(m_1) = a_2$ and $\mu(m_2) = a_1$. While this assignment is both depth optimal and stable, it is

[29]This is, of course, not to say that impartial arbiters have no preferences over matches. Both in case of sports and dispute resolution, arbiters might prefer some matches over others due to reasons unrelated to partiality. For example, a referee might prefer to handle more critical matches or an arbitrator, more publicly visible cases. The problem is, it is impossible to disentangle such motivation from others. Hence, both current real-life practices and the existing theoretical literature treats arbiters as objects rather than agents.

not fair over matches.

Overall, we observe that stability is incompatible with both depth optimality and fairness over matches.[30] An analysis of the trade-offs between their requirements as well as domain restrictions under which they become compatible are left for future research.

7 Concluding Remarks

We conclude with a discussion of possible extensions and issues relevant to practical applications. First, some disputes such as those in inheritance or bankruptcy may involve more than two agents. Our model has a straightforward extension to this case where matches can involve more than two agents, and the depth of an arbiter in a match is its worst ranking in the preferences of the agents involved in the match. All the other notions, including the mechanisms, remain the same. We already know that UC loses some of its properties once we go beyond the two-agent case. For instance, it can contain more than two arbiters, and UC arbiters are not necessarily ranked above the median (Brams and Kilgour, 2001). But fortunately, all of our results except Theorem 2 are easily generalizable. Their proofs go through almost without any modification.[31] The proof of Theorem 2, however, does not hold (see Footnote 33 for a detailed discussion). Whether the statement itself continues to hold is an open question. Second, in some sports, instead of a single head referee there might be multiple referees of equal standing. Similarly, some disputes might require more than one arbitrator. The extension of our model to these cases is left for future research.

We assume that agents have strict preferences. It is not trivial to extend our analysis to allow for indifference. For instance, an immediate approach could be to use a tie-breaking rule to obtain strict preferences and apply the current mechanisms. This method, however, cannot work as no mechanism, including DOM, can produce a depth optimal assignment purely based on these artificially obtained strict preferences.[32] While one could still modify DOM to make it depth optimal, we suspect that the modified mechanism would have to entail complicated reassignment cycles and chains to improve depth (e.g. as in Erdil and Ergin (2017)). It is also not clear whether the other results would carry over.

Some real life applications might incorporate constraints on which matches an arbiter can be assigned to. For instance, arbitral institutions might want to consider for a case only arbiters

[30] On the other hand, there are mechanisms that are both stable and strategy proof. For example, exogenously picking from every match one agent and using the deferred-acceptance mechanism where the selected agents make offers to the arbiters in decreasing order of their preferences satisfies both properties. Note that in the deferred-acceptance, an arbiter rejects an agent for the sake of another offering agent from more preferred matches.

[31] One needs to strengthen the third condition of the PIP axiom accordingly.

[32] For instance, consider $M = \{m_1, m_2\}$ and $A = \{a_1, a_2\}$, each with unit capacity and $m_1 \succ m_2$. Let ψ be a mechanism, which is only defined over strict preferences. Suppose that both agents in m_1 are indifferent between the arbiters, while a_1 is unanimously preferred by both agents in m_2. Let the tie-breaking rule be such that it favors the lower indexed arbiters. Thus, under the artificial preferences, a_1 is preferred to a_2 by each agent. If ψ assigns a_1 to m_1, the assignment cannot be depth optimal. On the other hand, if ψ assigns a_2 to m_1, then we can consider an alternative problem where agents in m_2 now prefer a_2. Since the tie-breaking is still the same, ψ continues to assign a_2 to m_1, violating depth optimality.

in "close proximity to it". Or in a bitter dispute or rivalry, some inexperienced arbiters can be excluded. Similarly, referees who did not perform well in the previous match-week (or in the previous match between the two teams) can be excluded. Such considerations can be easily included in our model by restricting the set of feasible assignments correspondingly. All the notions can be defined in terms of this smaller feasible set of assignments. And all of our analysis goes through.

We model arbiters as objects rather than agents. That is, we disregard any possible preferences they might have. This is consistent with the previous literature on dispute resolution and sports, which depicts arbiters as ideally impartial professionals providing a service, supporting our modeling choice. Nevertheless, we can incorporate in our model arbiters' preferences over matches (if any). Note that depth optimality solely depends on the agents' preferences. Hence, the arbiters' preferences can only be used to break ties among multiple depth optimal assignments. Recall that DOM already uses an exogenous tie-breaker. Making it depend on the arbiters' preferences would not affect any of our results. Finally, if needed, the arbiters' right to decline service (a form of binary preference relation) can be modeled as a constraint, as discussed in the previous paragraph.

In practical applications, a number of additional issues can arise. First, some agents may prefer not to participate in the mechanism or not publicly declare their preferences. As a potential solution, such matches can be left outside of the mechanism and later receive their arbiters through a public lottery. Second, FSD and $FSED$ pick the best arbiter of one agent in each match. This might raise fairness complaints. As a solution, an ex-ante lottery can be used. In some sports, teams play two matches, one at the home town of each. In such cases, for example the visiting team's best arbiter can be picked.

Finally, in dispute resolution the arbitral institute must decide on a time-window in which the arrived cases are collected. The decision depends on several factors such as the frequency with which cases arrive, their duration, as well as their urgency. Analyzing such a dynamic (and stochastic) extension of our model would be a natural next direction.

Appendices

A The Proofs of Propositions 1, 2, 3, 4, and Theorems 1 and 2

Unless there is danger of confusion, we suppress the preferences P in the depth notation and write $d_m(a)$ for the depth of arbiter a for match m at P.

Proof of Proposition 1. *[Depth optimality implies Pareto optimality]* Let μ be an assignment that is not Pareto optimal for P. Then, there is an alternative assignment μ' such that for each agent i, $\mu' \ R_i \ \mu$, where it strictly holds for some agent. Note that as both agents in the same match are affected by their match's arbiter change, for some match m and agents $i, j \in I(m)$, $\mu' \ P_i \ \mu$ and $\mu' \ P_j \ \mu$. Thus, $d_m(\mu'(m)) < d_m(\mu(m))$. Moreover, for each other match m',

$d_{m'}(\mu(m')) \geq d_{m'}(\mu'(m'))$. Hence, μ is not depth optimal for P.

[Pareto optimality does not imply depth optimality] Consider a problem P where for each $k \geq 3$ and $i \in I(m_k)$, $\gamma(P_i, a_k) = 1$, that is, a_k is the top arbiter of each agent $i \in I(m_k)$. Let each arbiter have unit capacity. Let the first two matches' agents have the following preferences:

P_{11}	P_{12}	P_{21}	P_{22}
a_1	a_2	a_1	a_1
a_2	a_1	a_2	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Assignment μ where $\mu(m_1) = a_1$, $\mu(m_2) = a_2$, and $\mu(m_k) = a_k$ for each $k \geq 3$ is Pareto optimal at P. But it is not depth optimal. To see this, note that $d_{m_1}(\mu(m_1)) = d_{m_2}(\mu(m_2)) = 2$. Consider an alternative assignment μ' where $\mu'(m_1) = a_2$, $\mu'(m_2) = a_1$, and $\mu'(m_k) = a_k$ for each $k \geq 3$. We have $d_{m_1}(\mu'(m_1)) = 2$, $d_{m_2}(\mu'(m_2)) = 1$, and $d_{m_k}(\mu'(m_k)) = d_{m_k}(\mu(m_k)) = 1$, yielding an improvement in terms of depth profile, showing the desired conclusion. □

Proof of Theorem 1. *[Every depth optimal assignment can be obtained as the outcome of a DOM under some ordering of matches and arbiters.]*

Let us first observe an easy fact. For any $A' \subseteq A$, match m, and problem P, if $a \in UC(m, A)$ and $a' \in UC(m, A') \setminus UC(m, A)$, then $d_m(a) < d_m(a')$.

Let μ be a depth optimal assignment at problem P. Let $B^1 = \{m \in M : \ \mu(m) \in UC(m, A)\}$. We now claim that $B^1 \neq \emptyset$. Suppose $B^1 = \emptyset$ and pick a match m. From our supposition, $\mu(m) \notin UC(m, A)$. If, for some $a \in UC(m, A)$, $\mu(a) = \emptyset$, then μ cannot be depth optimal. This is because we can define an alternative assignment, say μ', by letting match m receive a while keeping all the other assignments same as those under μ. From above, $d_m(\mu'(m)) < d_m(\mu(m))$, and for every $m' \neq m$, $d_{m'}(\mu'(m')) = d_{m'}(\mu(m'))$.

Let $a \in UC(m, A)$. By the previous paragraph, $a = \mu(m')$ for some $m' \neq m$. By our supposition, $m' \notin B^1$. Let $a' \in UC(m', A)$. Then by the previous paragraph, there is some match $m'' \neq m'$ such that $a' = \mu(m'')$. If $m'' = m$, then we can define an alternative assignment μ' from μ where matches m and m' swap their assignments while all other matches receive their arbiters under μ. But then, $d_m(\mu'(m)) < d_m(\mu(m))$, $d_{m'}(\mu'(m')) < d_{m'}(\mu(m'))$, and for all $m'' \notin \{m, m'\}$, $d_{m''}(\mu'(m'')) = d_{m''}(\mu(m''))$, contradicting depth optimality of μ. Alternatively, if $m'' \neq m$, we apply the same steps to m''. As there are finitely many matches, we eventually end up with a match that we have already considered before. In other words, we obtain a cycle of matches. We let each of these matches receive the next match's arbiter in order. By construction, each match m in the cycle start receiving an arbiter from their $UC(m, A)$. Since we keep the others' assignments the same as those under μ, we obtain an assignment with a lower depth profiles than μ, contradicting the assumption that μ is depth optimal. Thus, we conclude that $B^1 \neq \emptyset$.

Let us consider Δ, where at the problem's priority ordering $\succ$, it orders the matches such that the matches in B^1 come earlier than $M \setminus B^1$. Let us write – $m_1, .., m_k$ – for this ordering among the matches in B^1. Additionally, let the arbiter ordering be such that the assignments of the matches in B^1 come first, while the ordering among them follows their assigned matches' enumeration. That is, the ordering over these arbiters follows $\mu(m_1), ..., \mu(m_k)$.

Let us exclude the matches in B^1 along with their assignments under μ. Let us write A' and M' for the set of remaining arbiters and matches, respectively. Let $B^2 = \{m \in M' : \mu(m) \in UC(m, A')\}$. Following the above steps, it is straightforward to conclude that $B^2 \neq \emptyset$. We then repeat the same procedure for the matches in B^2. Once we continue in the same manner until no match is left, we eventually derive a match and an arbiter orderings. By construction, DOM under Δ gives μ.

[Every DOM produces depth optimal and Pareto optimal assignments.] Let us consider a problem P and Δ such that at the problem, it produces the orderings of $m_1, ..., m_n$ and $a_1, ..., a_r$. Consider the induced DOM. Let μ be its outcome at problem P. We claim that μ is depth optimal.

Let $B^1 = \{m \in M : \mu(m) \in UC(m, A)\}$. By the definition of DOM, $B^1 \neq \emptyset$. Let $m_{k'}$ be the last match (with respect to the ordering) such that $\mu(m_{k'}) \in UC(m_{k'}, A)$. If $k' = n$, then depth optimality follows from the definition of UC. If $k' < n$, let $A' = A \setminus \cup_{k=1}^{k'} \mu(m_k)$. Let $B^2 = \{m \in M : \mu(m) \in UC(m, A')\}$. By the definition of DOM, $B^2 \neq \emptyset$, and no match in B^2 can obtain an arbiter with a lower depth than that under μ while keeping all the matches in B^1 receiving their lowest depth arbiters. In other words, there is no way of decreasing the depth of some match in B^2 without increasing the depth of some match in B^1. If we exclude the arbiter assignments of the matches in B^2 from the problem and keep applying the same analysis, we easily conclude that for any $m \in B^3$, there is no way of assigning a lower depth arbiter to match m without increasing the depth of some match in $B^1 \cup B^2$. Once we invoke the same line of arguments to cover all matches, we eventually conclude that, under μ, there is no way of decreasing the depth of some match without increasing some other's. This shows that μ is depth optimal. From Proposition 1, μ is Pareto optimal as well. □

***Proof of Proposition** 2.* Let ψ be a DOP. Let P be a problem and $\psi(P) = \mu$. From Theorem 1, μ is depth optimal and hence from Proposition 1, μ is Pareto optimal. Suppose μ is not fair over matches. Then, there is a pair of matches m, m' such that $m \succ m'$ and $\mu(m') \; P_i \; \mu(m)$ for each agent $i \in I(m)$. Hence, $d_m(\mu(m')) < d_m(\mu(m))$. But $m \succ m'$ implies that match m comes before match m' in ψ. This in turn implies $d_m(\mu(m)) \leq d_m(\mu(m'))$, a contradiction. □

***Proof of Proposition** 3.* Consider a problem P where for each $k \geq 3$ and $i \in I(m_k)$, $\gamma(P_i, a_k) =$

1. Let each arbiter have unit capacity. The first two matches' agents have the following preferences:

P_{11}	P_{12}	P_{21}	P_{22}
a_1	a_2	a_2	a_1
a_2	a_1	a_1	a_2
a_3	a_3	a_3	a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let ψ be a depth optimal mechanism. Let $\psi(P) = \mu$. Note that for each $k \geq 3$, $\mu(m_k) = a_k$. We have the following two cases to consider.

Case 1. Suppose $\mu(m_1) = a_1$ and $\mu(m_2) = a_2$. Consider $P'_{12}:\ a_2, a_3, a_1$ and write $P' = (P'_{12}, P_{-12})$. At P', there is only one depth optimal assignment: $\mu'(m_1) = a_2$, $\mu'(m_2) = a_1$, and $\mu'(m_k) = a_k$ for each $k \geq 3$. Hence, $\psi(P') = \mu'$, showing that agent 12 benefits from misreporting.

Case 2. Suppose $\mu(m_1) = a_2$ and $\mu(m_2) = a_1$. Consider $P''_{11}:\ a_1, a_3, a_2$ and write $P'' = (P'_{11}, P_{-11})$. At P'', there is only one depth optimal assignment: $\mu''(m_1) = a_1$, $\mu''(m_2) = a_2$, and $\mu''(m_k) = a_k$ for each $k \geq 3$. Hence, $\psi(P'') = \mu''$, showing that agent 11 benefits from misreporting. □

Proof of Proposition 4. Let $M = \{m_1, ..., m_n\}$ and $A = \{a_1, ..., a_{n+1}\}$, each having a unit capacity. Let P be such that for each match m_k with $k > 1$, the set of the least preferred arbiters of the agents in m_k is $\{a_1, a_2\}$. Let P_{11} and P_{12} be as follows:

P_{11}	P_{12}
a_1	a_2
a_2	a_1
$\vdots$	$\vdots$

Let ψ be a minimally compromising mechanism. $MC(P) = \{\mu \in \mathcal{M} :\ \mu(m_k) \in A \backslash \{a_1, a_2\}$ for each $k > 1\}$. Hence $MC(P) \neq \emptyset$. Let us write $\psi(P) = \mu$. As $\mu \in MC(P)$, we have $\mu(m_1) \in \{a_1, a_2\}$.

Case 1. Suppose $\mu(m_1) = a_1$. Let us consider $P'_{12}:\ a_2, ..., a_1$, and write $P' = (P'_{12}, P_{-12})$. $MC(P')$ is singleton, including μ' where $\mu'(m_1) = a_2$. Thus, agent 12 benefits from misreporting.

Case 2. Suppose $\mu(m_1) = a_2$. Let us consider $P''_{11}:\ a_1, ..., a_2$, and write $P'' = (P''_{11}, P_{-11})$. $MC(P'')$ is singleton, including μ'' where $\mu''(m_1) = a_1$. Thus, agent 11 benefits from misreporting. Hence, ψ cannot be strategy-proof concluding the proof. □

Proof of Theorem 2. Let ψ and ϕ be two different DOM. Whenever their used match-arbiter orderings are the same at $\succ$, then both of them produce the same assignments for each P. Therefore, these orderings differ for at least some $\succ$. In what follows, we consider any $\succ$ where they differ.

Case 1: [Suppose that the match orderings are different] Suppose that the match orderings of ψ and ϕ are the same for the first $j-1$ matches. Let m and m' be the matches coming in the j^{th} place in the orderings under ψ and ϕ, respectively.

Consider a problem where $q_a = 1$ for each $a \in A$. Construct a preference profile P where (i) for each match $m_k \in M \setminus \{m, m'\}$ and agent $i \in I(m_k)$, $\gamma(P_i, a_k) = 1$, where $a_k \neq a_{k'}$ for any $k \neq k'$, (ii) for each agent $i \in I(m)$, $\gamma(P_i, a) = 1$, and (iii) for $i, j \in I(m')$, $P_i: \ a, a', ...$; and $P_j: \ a', a, ...$; where a, a' are different from each top arbiter of the agents in $I(M \setminus \{m, m'\})$. For ease of notation, let $\psi(P) = \mu$ and $\phi(P) = \mu'$.

At μ, each match $m_k \in M \setminus \{m, m'\}$ receives the common top arbiter a_k of its agents. This, as well as the order of agents above implies that $\mu(m) = a$ and hence, $\mu(m') = a'$. Note that every agent except those in m' receive their top arbiters. Agent j in match m' also receives her top arbiter, r'. Agent $i \in I(m')$ receives her second choice, however, as arbiter a is the top common arbiter of the match m, which is coming before m' in the match ordering under ψ, there is no way for agent i to obtain arbiter a through misreporting. All these imply that ψ is not manipulable at P.

At μ', each match $m_k \in M \setminus \{m, m'\}$ receives the common top arbiter of its agents as well. However, as m' comes earlier than m in the ordering under ϕ, we have $\mu'(m') \in \{a, a'\}$. Without loss of generality, let $\mu'(m') = a'$. Recall that $i \in I(m')$ and $a \ P_i \ a'$. Consider P_i' which is identical to P_i except a' is now ranked at the bottom, while its original (second) rank under P_i is taken by the worst arbiter of the other agent $j \in I(m')$ under P_j. Note that because of our supposition that $|A| \geq 3$, $P_i' \neq P_i$. Let $P' = (P_i', P_{-i})$. By construction, we have $UC(m, P') = \{a\}$. Thus, at $\phi(P')$, match m is assigned arbiter a, showing that agent i benefits from misreporting. Hence, ψ cannot be more manipulable than ϕ.

On the other hand, we can straightforwardly define a similar problem, say P'', having the same features as above, but now $UC(m, P'') = \{a, a'\}$, $UC(m', P'') = \{a\}$. A similar analysis will then show that ϕ is not manipulable at P''. Yet, ψ is manipulable by an agent in match m. Hence, ϕ is not more manipulable than ψ.

Case 2: [ψ and ϕ differ in the ordering of arbiters but not in the order of matches.] Let us suppose that the match orderings are the same under both ψ and ϕ, but their arbiter orderings are different. Let P be a problem where $\psi(P) \neq \phi(P)$. Let us continue writing μ and μ' for their outcomes, respectively. It means that for some match m, $\mu(m) \neq \mu'(m)$. Let $\mu(m) = a$ and $\mu'(m) = a'$. As the match orderings are the same, under both mechanisms, match m receives an arbiter from $UC(m, P, A')$ from some $A' \subseteq A$. Therefore, we have $\{a, a'\} \subseteq UC(m, P, A')$. Moreover, as $UC(m, P, A')$ can contains at most two arbiters (Brams and Kilgour (2001)), we have $UC(m, P, A') = \{a, a'\}$.

Let $i, j \in I(m)$. Without loss of generality, let $a \ P_i \ a'$ and $a' \ P_j \ a$ (note that if both were to prefer a (a') to a' (a), then a' (a) could not have been included in $UC(m, P, A')$). Consider

P'_j that preserves the ordering under P_j except it puts a at the end of the list while replacing its original position with the worst arbiter of agent i under P_i. Let $P' = (P'_j, P_{-j})$. At problem P', all the agents but agent j have the same preferences. This implies that under both mechanisms, match m continue receiving an arbiter from $UC(m, P', A')$ at problem P'. However, in this case, we have $UC(m, P', A') = \{a'\}$.[33] This means that match m receives a' at P' under both mechanism' outcomes, showing that agent j beneficially manipulates ψ at P.

On the other hand, from above, we have $a \; P_i \; a'$. Hence, from a symmetric argument, agent i can manipulate ϕ at P. Thus, at any problem P where $\psi(P) \neq \phi(P)$, both are manipulable.

Next, consider a problem P where $\psi(P) = \phi(P)$. Suppose ψ is not manipulable at P. If ϕ is not manipulable at P, then the proof is done. Otherwise, let us suppose that ϕ is manipulable at problem P. That is, for some agent $i \in I(m)$ and P'_i, $\phi_m(P'_i, P_{-i}) \; P_i \; \phi_m(P)$. Let $\phi_m(P) = a$ and $\phi_m(P'_i, P_{-i}) = a'$.

Let $m_k \in M$. At problem P, let ϵ^n be the set of assignments from which an arbiter is assigned to m_k in ψ. We now claim that ϵ^n cannot contain a pair of assignments assigning different arbiters to match m_k. Assume for a contradiction that m_k is assigned to a and a' at some assignments in ϵ^n. If we write A' for the set of arbiters having a positive reduced capacity after excluding the those previously assigned, then $a, a' \in UC(m_k, P, A')$. This implies that a is preferred to a' by one of the agents in m_k, while the converse is true for the other agent in m_k. Because, otherwise, that is, whenever either of them, say a, is preferred to a' by both agents in m_k, arbiter a' cannot belong to $UC(m_k, P, A')$.

Without loss of generality, let $\psi_{m_k}(P) = a$ and $a' \; P_i \; a$ for $i \in m_k$. Let P'_i be a false preference list where it preserves the ordering under P_i except it puts a at the end of the list while replacing its original position with the worst arbiter of the other agent $j \in I(m_k)$. Let us write $P' = (P'_i, P_{-i})$. Note that at P', ψ remains working the same as at P until the step where match m_k receives an assignment. Thus, A', from which an arbiter is assigned to m_k, is the same at P and P'. However, we now have $UC(m_k, P', A') = \{a'\}$, hence $\psi_{m_k}(P') = a'$, showing that agent i can benefit by misreporting. This, however, contradicts our supposition that ψ is not manipulable at P.

Therefore, there is no multiplicity in the arbiter assignments in ψ at P, in other words, the arbiter ordering does not have any role in determining the outcome of ψ. As the only difference between ψ and ϕ is the arbiter ordering, the same is true for ϕ. That is, in ϕ, no match m's arbiter assignment is found by using the arbiter ordering (that is, tie-breaking). Therefore, agent i cannot manipulate ϕ at problem P through reverting the arbiter selection.

Let A' be the set of arbiters having a positive reduced capacity after removing the assignments of the matches coming before m in ϕ. Note that it is the same under ψ as well (because $\psi(P) = \phi(P)$, and the match ordering is the same under both mechanisms). The fact that agent i benefits from reporting P'_i under ϕ yields two possible cases: at (P'_i, P_{-i}), (i) either a' and a'' (a'' may the same

[33] This argument does not work whenever matches contain more than two agents. To see this, let us consider a match m, consisting of three agents i, j, k. Let $A = \{a_1, a_2, a_3\}$, with P_i : a_1, a_2, a_3; P_j : a_2, a_3, a_1; P_k : a_3, a_1, a_2. We have $UC(m, A) = \{a_1, a_2, a_3\}$, and agent i does not have a false reporting making the UC singleton, including either a_1 or a_2. The symmetric arguments are true for the other agents as well.

as a) are available to be assigned to match m, and the arbiter ordering under ϕ causes a' to be assigned, or (ii) a' comes to be the only alternative that can be assigned to match m. If the latter is the case, then ψ would be manipulable at P as well, contradicting our supposition. Hence, the former is the case. Because of the non-manipulability of ψ at P, a'' is assigned to match m (because of the arbiter ordering under ψ), and $a \; R_i \; a''$.

Let us next consider a problem P'' where, for each agent j, P''_j is the same as P_j except the positions of the arbiters a' and a'' are swapped. Let us also swap the capacities of a and a'. We obtain a symmetric problem to P. This, as well as the fact that the arbiter ordering does not matter at problem P (from above), we have $\psi(P'') = \phi(P'')$, hence $\psi_m(P'') = a$. Because of the tie-breaking rule, at P'', ϕ cannot be manipulated by agent i through reporting P'_i above. However, ψ becomes manipulable as $a'' = \psi_m(P'_i, P_{-i}) \; P''_i \; r$.

Hence, we have a problem P'' where ψ is manipulable by agent i through reporting P'_i, while it is not true under ϕ. If we repeat the above exercise for each other agent along with its beneficial misreporting under ϕ at problem P, we eventually obtain a problem at which ψ is manipulable, whereas ϕ is not. Hence, neither of ψ or ϕ is more manipulable than the other. □

B Proofs of Proposition 5, Lemma 1, Corollary 3

Proof of Proposition 5. In $FSED$, no agent can affect the assignments of the matches coming earlier than theirs. Moreover, no agent in a match can alter the match's decisive agent (i.e. the agent who picks her top remaining arbiter). All these imply that each $FSED$ is strategy-proof. On the other hand, as π respects $\succ$, higher priority matches always come earlier than lower priority ones in the mechanism. This, in turn, implies that the agents of a higher priority match never unanimously prefer the arbiter assignment of a lower priority match. Hence, each $FSED$ is fair over matches. □

Proof of Lemma 1. Let ψ be a one-sided responsive and non-bossy mechanism. Because of its ones-sided responsiveness, for each match m, there is an agent $i \in I(m)$ such that $\psi_m(P) = \psi_m(P')$ where $P_i = P'_i$. Let $I' = \{i_1, ..., i_n\}$ be the set of these agents, where $i_k \in I(m_k)$ for each $k \leq n$. Let P and P' be preference profiles such that $(P_i)_{i \in I'} = (P'_i)_{i \in I'}$. Let $i_1 \in I'$ and $j_1 \in I(m_1) \setminus \{i_1\}$. Let $P^1 = (P'_{j_1}, P_{-j_1})$. By one-sided responsiveness of ψ, we have $\psi_{m_1}(P) = \psi_{m_1}(P^1)$. By the non-bossiness of ψ, $\psi(P) = \psi(P^1)$. Let $j_2 \in I(m_2) \setminus \{i_2\}$ and $P^2 = (P'_{j_2}, P^1_{-j_2})$. By one-sided responsiveness of ψ, we have $\psi_{m_2}(P^1) = \psi_{m_2}(P^2)$. By the non-bossiness of ψ, $\psi(P^1) = \psi(P^2)$. Hence, $\psi(P) = \psi(P^1) = \psi(P^2)$. If we continue applying the same analysis for the other j_k in matches m_k with $k > 2$, we eventually obtain $P^n = P'$ and $\psi(P) = \psi(P')$. □

Before proceeding with the next proof, we define neutrality. To this end, let $\sigma : A \to A$ be a permutation of arbiters; and for each P, let P^σ be such that for each agent i, $a \; P_i \; b$ if and only if $\sigma(a) \; P^\sigma_i \; \sigma(b)$. A mechanism ψ is **neutral** if for each problem P, match m, and permutation σ, $\sigma(\psi_m(P)) = \psi_m(\sigma(P))$.

Proof of Corollary 3. Let ψ be a mechanism that satisfies the stated axioms. By Lemma 1, only the preferences of a specific subset of agents (one agent from each match) matter. Hence, the mechanism class we consider reduces to one in standard assignment problems, where each match consists of a single agent. This and Svensson (1999) together imply that ψ is an SD. Moreover, as ψ is fair over matches, for each $\succ$, the match ordering under ψ has to respect $\succ$, hence it is an FSD. It is immediate to see that each FSD satisfies the properties. $\square$

- **Independence of the Axioms in Corollary 3**

Let us consider a mechanism that works as the same as FSD but under some Γ where Γ does not respect $\succ$. It satisfies all the properties except fairness over matches. Let $M = \{m_1, m_2\}$ and $A = \{a_1, a_2, a_3\}$, with $m_1 \succ m_2$. Let ψ be a mechanism such that match m_1 always receives the top arbiter of a specific agent $i_1 \in I(m_1)$. Among the remaining arbiters, if both agents in m_2 have the same top alternative, then match m_2 is assigned to that arbiter. Otherwise, match m_2 receives the worst remaining arbiter of a specific agent $i_2 \in I(m_2)$. For each other problem, consisting of different sets of matches, or arbiters, or the priority ordering, let ψ be equivalent to some FSD. This mechanism satisfies all the properties except one-sided responsiveness.

Let $M = \{m_1, m_2, m_3\}$ and $A = \{a_1, a_2, a_3\}$, with $m_1 \succ m_2$ and $m_2 \sim m_3$. Let Γ and Γ' be such that $\Gamma_\succ = \{(m_1, i_{12}), (m_2, i_{22}), (m_3, i_{32})\}$ and $\Gamma'_\succ = \{(m_1, i_{12}), (m_3, i_{32}), (m_2, i_{22})\}$. Let us consider a mechanism ψ such that it runs FSD under $\Gamma_\succ$ ($\Gamma'_\succ$) whenever the top choices of the agents in m_1 are (not) the same. For each other problem, ψ produces the same assignments as FSD under Γ. This mechanism satisfies all the properties except non-bossiness.

Let $M = \{m_1, m_2\}$, where $m_1 \sim m_2$, and $A = \{a_1, a_2, a_3\}$. Let ψ be a mechanism such that at each problem, m_1 receives the top arbiter of agent i_{12}, and m_2 receives the least preferred arbiter of agent i_{22} among the remaining ones. For each other problem, ψ is equivalent to some FSD. This mechanism satisfies all the properties except strategy-proofness.

In the same economy above, let Γ and Γ' be such that let $\Gamma_\succ = \{(m_1, i_{11}), (m_2, i_{21})\}$ and $\Gamma'_\succ = \{(m_2, i_{21}), (m_1, i_{11})\}$. Consider a mechanism ψ that runs FSD under $\Gamma_\succ$ ($\Gamma'_\succ$) whenever the top arbiter of i_{11} is (not) a_1. For each other problem, ψ is nothing but FSD under Γ. This mechanism satisfies all the properties except neutrality.

C Proofs of Theorems 3 and 4

Let us write $\mathbb{G}$ for the of match-agent orderings respecting $\succ$. The following lemma will be critical in the proof of Theorem 3.

Lemma 2. *Let ψ be a mechanism that is PIP, fair over matches, and strategy-proof. Then, at each problem P, there exists some $\Upsilon \in \mathbb{G}$ such that $\psi(P) = FSD(P)$ under Υ. Moreover, Υ remains unchanged across quota profiles so long as the preferences are the same.*

Proof. Let $\psi(P) = \mu$. Since ψ is PIP, at problem P, there is a match-agent ordering $\Upsilon' = \{(m_1, i_1), ..., (m_n, i_n)\}$ such that for each i_k and P'_{i_k}, $\psi_{m_{k'}}(P) = \psi_{m_{k'}}(P'_{i_k}, P_{-i_k})$ for each $k' < k$, and $\gamma(P'_{i_k}, \psi_{m_k}(P'_{i_k}, P_{-i_k}), A') = \gamma(P_{i_k}, \psi_{m_k}(P), A')$ where $A' = A \setminus \{a \in A : \; q_a = |\{k' < k : \psi_{m_{k'}}(P) = a\}|\}$.

We now claim that $\psi_{m_1}(P)$ is the top arbiter of agent i_1, say a. Assume it is not the case. This means that $\gamma(P_{i_1}, \psi_{m_1}(P), A) = k > 1$. Let P'_{i_1} be such that arbiter a has the k^{th} ranking under P'_{i_1}. Note that $P_{i_1} \neq P'_{i_1}$ as the arbiter a is the top choice under P_{i_1}, whereas she is the k^{th} top choice, where $k > 1$, under P'_{i_1}. By PIP of ψ, we have $\gamma(P'_{i_1}, \psi_{m_1}(P'_{i_1}, P_{-i_1}), A) = k$, implying that $\psi_{m_1}(P'_{i_1}, P_{-i_1}) = a$. This, however, shows that agent i_1 benefits from reporting P'_{i_1} at problem P, contradicting the strategy-proofness of ψ. Therefore, $\psi_{m_1}(P)$ is the top arbiter of i_1. If we apply a symmetric analysis for i_2, we easily conclude that $\psi_{m_2}(P)$ is the top arbiter of agent i_2 among those in A' where $A' = A \setminus \{a\}$ if $q_a = 1$, and otherwise, $A' = A$. If we apply the same analysis for the other agents in order, we conclude that $\psi(P) = FSD(P)$ under Υ'. If Υ' respects $\succ$, then $\Upsilon' \in \mathbb{G}$, hence the proof is done. Otherwise, we need to come up with $\Upsilon \in \mathbb{G}$ such that $\psi(P) = FSD(P)$ under Υ.

Assume for a contradiction that there is no $\Upsilon \in \mathbb{G}$ such that $\psi(P) = FSD(P)$ under Υ. From above, we know that $\psi(P) = FSD(P)$ under Υ'. These imply that for some matches m_k and $m_{k'}$, we have $k' < k$ (that is, $m_{k'}$ comes before m_k at Υ'), $m_k \succ m_{k'}$, and at $\psi(P)$, $m_{k'}$ receives an arbiter that is preferred by agent i_k. Let $\psi_{m_{k'}}(P) = a$. Let $j_k \in I(\{m_k\}) \setminus \{i_k\}$. Let P'_{j_k} be such that arbiter a is the top choice of agent j_k. By PIP of ψ, we have $\psi_{m_{k''}}(P'_{j_k}, P_{-j_k}) = \psi_{m_{k''}}(P)$ for each $k'' \leq k$; therefore, in particular, $\psi_{m_{k'}}(P'_{j_k}, P_{-j_k}) = a$ and $a \; P_{i_k} \; \psi_{m_k}(P'_{j_k}, P_{-j_k}) = \psi_{m_k}(P)$. Thus, at (P'_{j_k}, P_{-j_k}) and under $\psi(P'_{j_k}, P_{-j_k})$, both agents $i_k, j_k \in I(m_k)$ prefer arbiter a to their assignment, while match $m_{k'}$ obtains arbiter a. As $m_k \succ m_{k'}$, it implies that ψ cannot be fair over matches, yielding a contradiction. Note that the analysis is independent of the quotas. Hence, $\psi(P) = FSD(P)$ under Υ for any q. □

Proof of Theorem 3. *["Only If" Part.]* Let us consider $q_a = 1$ for each $a \in A$. Let P be such that for each agent i, $P_i : \; a_1, ..., a_r$. By Lemma 2, $\psi(P) = FSD(P)$ under some $\Upsilon' \in \mathbb{G}$. This implies that the arbiters $a_1, ..., a_n$ are assigned to the matches at $\psi(P)$ (note that as $q_a = 1$ for each $a \in A$, $r \geq n$). Without loss of generality, let $\psi_{m_k}(P) = a_k$ for each $k \in \{1, .., n\}$. This also implies that under Υ', the matches come in the order of $m_1, ..., m_n$. By PIP, at problem P, there exists a match-agent ordering, say Υ, such that PIP's conditions hold. Let $\Upsilon = \{(m'_1, i_1), .., (m'_n, i_n)\}$.

We claim that $m'_k = m_k$ for each $k \in \{1, .., n\}$. For each $k \in \{1, .., n\}$, let $j_k \in I(m_k)$. We have $\gamma(P_{j_1}, \psi_{m_1}(P), A) = 1$. Hence, by PIP, for some $i_1 \in I(m_1)$, $\gamma(P'_{i_1}, \psi_{m_1}(P'_{i_1}, P_{-i_1}), A) = 1$ for each P'_{i_1}. This implies that agent i_1 can affect each other match's assignment by changing her preferences, implying that $m_1 = m'_1$. On the other hand, we have $\gamma(P_{j_2}, \psi_{m_2}(P), A') = 1$, where $A' = A \setminus \{\psi_{m_1}(P)\}$ (recall that $q_a = 1$ for each $a \in A$). By the same token as above, for some $i_2 \in I(m_2)$ and each P'_{i_2}, $\gamma(P'_{i_2}, \psi_{m_2}(P'_{i_2}, P_{-i_2}), A') = 1$, implying that match $m'_2 = m_2$. If we continue in the same manner, we eventually conclude that $m'_k = m_k$ for each k.

We next find the agents in Υ. For this purpose, for some $i \in I(m_1)$, let $P'_i : \; a_2, ..., a_1$. If $\psi_{m_1}(P'_i, P_{-i}) = a_2$, then by PIP, $i_1 = i$. Otherwise, the other agent in m_1 constitutes i_1. Without loss of generality, let $i_1 = i$. Let $j \in I(m_2)$ and consider $P'_j : \; a_1, a_3, ..., a_2$. By PIP,

$\psi_{m_1}(P'_j, P_{-j}) = a_1$. If $\psi_{m_2}(P'_j, P_{-j}) = a_3$, then $i_2 = j$. Otherwise, the other agent in m_2 constitutes i_2. Once we repeat the same exercise for the other matches, we completely find out the agents in Υ. Without loss of generality, let $\Upsilon = \{(m_1, i_1), .., (m_n, i_n)\}$. Recall that at problem P, $\psi_{m_k}(P) = a_k$ for each $k \in \{1, .., n\}$. As, at P, the agents' preferences are the same, always preferring the arbiters with the lower indexes, we have $\psi(P) = FSD(P)$ under Υ.

Let P' be any problem. We continue keeping the unit capacities. In what follows, we show that $\psi(P') = FSD(P')$ under Υ. Let us define $P^1 = (P'_{i_1}, P_{-i_1})$. By PIP, $\gamma(P'_{i_1}, \psi_{m_1}(P^1), A) = 1$. That is, agent i_1 receives her top arbiter. Let $\psi_{m_1}(P^1) = a$. Let j_1 be the other agent in m_1 and define $P'^1 = (P'_{j_1}, P'_{i_1}, P_{-\{i_1,j_1\}})$. By PIP, $\psi_{m_1}(P'^1) = a$.

By consistency of ψ, for each $k \neq 1$, $\psi_{m_k}(P'^1|M \setminus \{m_1\}, q') = \psi_{m_k}(P'^1)$, where $q'_a = 0$ and $q'_{a'} = 1$ for each $a' \neq a$. Let $\psi(P'^1|M \setminus \{m_1\}, q') = \mu$. From above, $\mu = FSD(P'^1)$ under the ordering of $(m_2, i_2), ...(m_n, i_n)$ (in the reduced economy, consisting of $M \setminus \{m_1\}$ and q').[34] That is, these agents, one by one following the ordering, receive their top choices in the reduced problem. Moreover, from the same arguments above, at P'^1 (in the reduced problem), PIP holds for Υ.

We then turn to m_2. Let $P^2 = (P'_{i_2}, P'^1_{-i_2})$. By PIP, $\gamma(P'_{i_2}, \psi_{m_2}(P^2), A') = 1$ where $A' = A \setminus \{a\}$. Let $\psi_{m_2}(P^2) = a'$. Let j_2 be the other agent in m_2 and define $P'^2 = (P'_{i_2}, P'_{j_2}, P'^1_{-\{i_2,j_2\}})$. By PIP, $\psi_{m_2}(P'^2) = a'$.

By the consistency of ψ, for each $k \neq 2$, $\psi_{m_k}(P'^2|M \setminus \{m_2\}, q'') = \psi_{m_k}(P'^2)$, where $q''_{a'} = 0$ and $q''_a = 1$ for each other arbiter a. Let $\psi(P'^2|M \setminus \{m_2\}, q'') = \mu'$. From above, $\mu' = FSD(P'^2)$ under the ordering of $(m_1, i_1), (m_3, i_3), ...(m_n, i_n)$ (in the reduced economy, consisting of $M \setminus \{m_2\}$ and q''). Hence, $\psi(P'^2) = FSD(P'^2)$ under Υ. Moreover, PIP holds for Υ (in the reduced problem)

If we continue applying the same arguments till we obtain P', we eventually conclude that $\psi(P') = FSD(P')$ under Υ. Let Γ be such that for $\succ$, $\Gamma_\succ = \Upsilon$. By our supposition, Γ is independent of the capacity profile. By Lemma 2, for any quota profile, $\psi(P') = FSD(P')$ under $\Gamma_\succ$. Hence, $\psi(P') = FSD(P')$ under $\Gamma_\succ$ for each P' and quota profile. Recall that we already obtained that the ordering of the matches is the same under both Υ and Υ'. As $\Upsilon' \in \mathbb{G}$, we also have $\Upsilon \in \mathbb{G}$. Note that the analysis holds for an arbitrary $\succ$, hence we can define Γ for each $\succ$ as above. Therefore, ψ is nothing but a FSD under Γ.

["If" Part.] It is immediate to see that FSD under any Γ satisfies all the properties.

□

Proof of Theorem 4. *["Only If" Part.]* Let us consider $q_a = 1$ for each $a \in A$. Let P be a problem such that for each agent i, $P_i : \ a_1, ..., a_r$. By Lemma 2, $\psi(P) = FSD(P)$ under some $\Upsilon = \{(m_1, i_1), ...(m_n, i_n)\}$. By following the same arguments in the proof of Theorem 3, we conclude that at problem P and under ψ, PIP's conditions hold for Υ. Note that $\psi_{m_k}(P) = a_k$ for each $k \leq n$.

Let P' be any problem. We continue assuming the unit capacity profile. Let us pick some $k \in \{1, ..., n-1\}$. Let us consider P'' where, for each $k' > k$, and agent $i \in I(m_{k'})$, $P''_i = P'_i$. For each other agent j, $P''_j = P_j$. If we invoke PIP once for each agent in every match $m_{k'}$ for $k' > k$,

[34] The match-agent ordering in FSD in the reduced problem follows Υ (see Footnote 25 for the supposition).

we find that for each $k' \leq k$, $\psi_{m_{k'}}(P'') = a_{k'}$. Let $i_k, j_k \in I(m_k)$. By PIP, $\gamma(P''_{i_k}, \psi_{m_k}(P''), A') = 1$ where $A' = A \setminus \cup_{k'<k} a_{k'}$. Let $\tilde{P} = (P'_{i_k}, P'_{j_k}, P''_{-\{i_k,j_k\}})$. By PIP, $\psi_{m_{k'}}(\tilde{P}) = a_{k'}$ for each $k' < k$. Moreover, $\gamma(\tilde{P}_{i_k}, \psi_{m_k}(\tilde{P}), A') = 1$, where $A' = A \setminus \cup_{k'<k} a_{k'}$. That is, agent i_k continues receiving its top arbiter in A'. Notice that these arguments hold for any $k \in \{1, ..., n-1\}$. Thus, we conclude that the assignment of match m_k is never affected by the preferences of the agents coming after m_k in Υ. This implies that $\psi(P')$ is the $FSED$ outcome under some $\pi_{\succ}$, where $\succ$ is the priority ordering in the problem. Both π and Υ (by Lemma 2) are independent of the quota profile. Moreover, the analysis holds for an arbitrary $\succ$. Therefore, we can apply the same steps for any $\succ$ and reach the same conclusion. Therefore, ψ is $FSED$ under some π.

["If" Part.] It is easy to verify that each $FSED$ under any π satisfies all the properties.

□

D Independence of the Axioms in Theorem 3 and Theorem 4

Let us consider a FSD under Γ that does not respect $\succ$. It satisfies all the properties except fairness over matches. Let $A = \{a_1, a_2, a_3\}$, each with a unit capacity. Let $M = \{m_1, m_2, m_3\}$, with $m_1 \succ m_2$, $m_1 \succ m_3$, and $m_2 \sim m_3$. Let $\Upsilon = \{(m_1, i_{12}), (m_2, i_{22}), (m_3, i_{32})\}$ and $\Upsilon' = \{(m_1, i_{12}), (m_3, i_{32}), (m_2, i_{22})\}$. Let us consider a mechanism ψ such that it runs FSD under Υ whenever a_1 is the top arbiter of i_{11}. Otherwise, ψ runs FSD under Υ'. In any other problem, consisting of different set of matches, or arbiters, or the priority ordering, let ψ produce the same outcome as FSD under some Γ. Note that ψ is a $FSED$, but not FSD, as the ordering of m_2 and m_3 does change depending on the preferences of i_{11}. ψ satisfies all the properties except consistency.

Let $M = \{m_1, m_2\}$, where $m_1 \succ m_2$. Let ψ be a mechanism such that at each problem, m_1 receives the top arbiter of agent i_{12}, and m_2 receives the least preferred arbiter of agent i_{22} among the remaining ones. In any other problem, it runs FSD under some Γ. This mechanism satisfies all the properties except strategy-proofness.

Let $M = \{m_1, m_2, m_3\}$ with $m_1 \succ m_2 \succ m_3$. Let $A = \{a_1, .., a_4\}$, each with a unit quota. Let ψ be a mechanism such that agent i_{11} picks her top arbiter, and then i_{22} picks her favorite remaining arbiter. In the remaining set of arbiters, if the agents in m_3 have a common favorite arbiter, then this arbiter is assigned to m_3. Otherwise, i_{31} picks her favorite remaining arbiter. In any other problem, ψ is FSD under some Γ. This mechanism satisfies all the properties except PIP as both agents in m_3 can affect their assignment.

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